The largest real root of the independence polynomial of a unicyclic graph

Ben Cameron

University of Guelph/The King's University

ben.cameron@kingsu.ca

(Joint work with Iain Beaton)

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Independence Polynomial

The independence polynomial of a graph G, denoted $I(G,x)$, is the generating polynomial for the number of independent sets of G of each order.

$$I(G,x) = \sum_{k=0}^{\alpha(G)} i_k^G x^k.$$

(i_k^G denotes the number of independent sets of order k, $\alpha(G)$ denotes the independence number)

Proposition (Gutman-Harary 1983): If G is a graph and $v \in V(G)$, then

$$I(G,x) = I(G-v,x) + x \cdot I(G-N[v],x).$$

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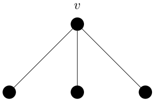

(a) S_4

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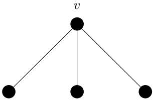

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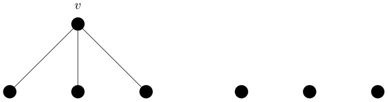

(a) S_4 (b) $S_4 - v$

$$I(S_4,x) = (1+x)^3 + x \cdot 1$$

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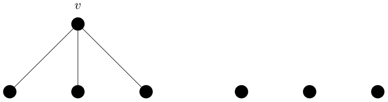
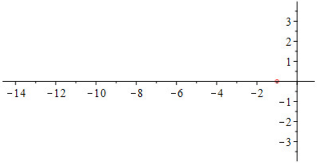


(a) S_4 (b) $S_4 - v$

$$\begin{aligned} I(S_4,x) &= (1+x)^3 + x \cdot 1 \\ &= 1 + 4x + 3x^2 + x^3 \end{aligned}$$

The roots of $I(G, x)$ are called the independence roots of G.



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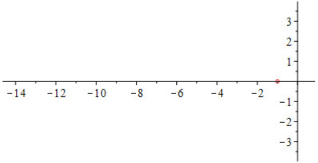

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Theorem (Brown-Hickman-Nowakowski 2004): The set of all independence roots is dense in $\mathbb{C}$.

Definition: Let $\beta(G)$ denote the independence root of G of smallest modulus.

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Bounds on β are open for all other families of graphs.

Definition (Csikvári 2013):

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Proposition (Csikvári 2013)

1) $\preceq$ is transitive.
2) If $H \preceq G$, then $\beta(H) \leq \beta(G)$.
3) If H is a subgraph of G, then $H \preceq G$.

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Theorem (Csikvári 2013) If T is a tree of order n, then $P_n \preceq T \preceq S_n$.

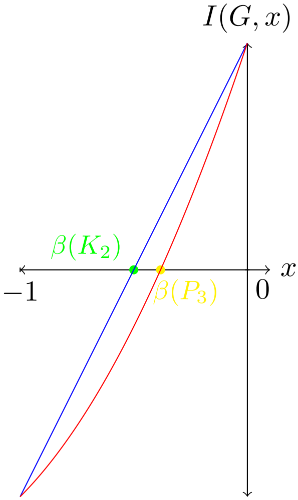

Figure: $K_2 \preceq P_3$ so $\beta(K_2) \leq \beta(P_3)$.

Question (Oboudi 2018): What are the maximal and minimal elements with respect to $\preceq$ among the family of connected unicyclic graphs?

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Theorem (Oboudi 2018): Let G and H be graphs with $u \in V(G)$, $e \in E(G)$, $v \in V(H)$, and $f \in E(H)$. Then the following hold:

(*i*) If $H - v \preceq G - u$ and $G - N[u] \preceq H - N[v]$, then $H \preceq G$.

(*ii*) If $H - f \preceq G - e$ and $G - N[e] \preceq H - N[f]$, then $H \preceq G$.

Theorem (Beaton-C. 2020+): If G is a connected unicyclic graph of order n, then $C_n \preceq G \preceq U_n$.

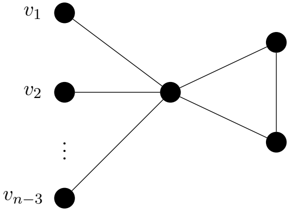
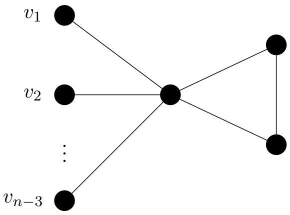


Figure: The graph U_n

Theorem (Beaton-C. 2020+): If G is a connected unicyclic graph of order n, then $C_n \preceq G \preceq U_n$.

Corollary: If G is a connected unicyclic graph of order n, then

$$\beta(C_n) \leq \beta(G) \leq \beta(U_n).$$



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Theorem (Brown-Nowakowski. 2001): Among all well-covered graphs of order n, the maximum modulus of an independence root is less than n.

Graph Star Operation

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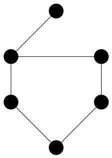

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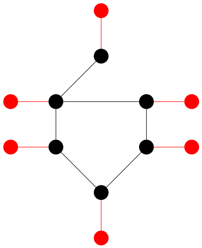

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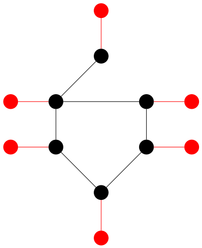

Figure: G^*.

Theorem (Topp-Volkmann 1990): G^* is well-covered for all graphs G.

Theorem (Finbow-Hartnell-Nowakowski 1993): If $G \neq K_1$ and $G \neq C_7$ and $\text{girth}(G) \geq 6$, then G is well-covered if and only if $G = H^*$.

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Lemma (Beaton-C. 2020+): Let G and H be graphs of order n. Then $H \preceq G$ if and only if $H^* \preceq G^*$.

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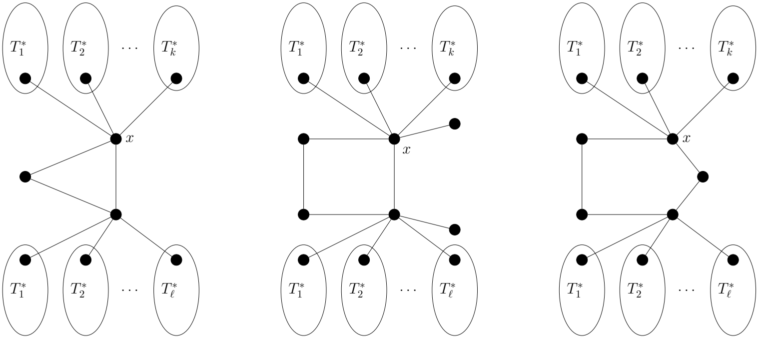
Lemma (Beaton-C. 2020+): Let G and H be graphs of order n. Then $H \preceq G$ if and only if $H^* \preceq G^*$.

Corollary: If T is a well-covered tree of order $2n$, then $P_n^* \preceq T \preceq S_n^*$.

Theorem (Topp-Volkmann 1990): A graph G is a connected well-covered unicyclic graph if and only if

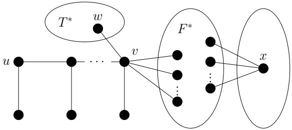
$$G \in \{C_3, C_4, C_5, C_7\} \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{KU}.$$



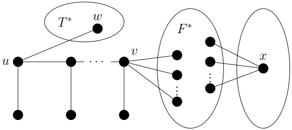
(a) $\mathcal{S}_3$ (b) $\mathcal{S}_4$ (c) $\mathcal{S}_5$

Note: All graphs in $\mathcal{S}_3$ and $\mathcal{S}_5$ have odd order while all graphs in $\mathcal{S}_4$ and $\mathcal{KU}$ have even order by definition.

The Dagger Operation

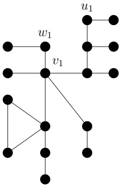


(a) G



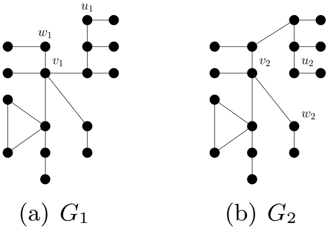
(b) $G^{\dagger}_{u,v,w}$

Lemma (Beaton-C. 2020+) If G is a graph as above, then $G^{\dagger}_{u,v,w} \preceq G$.



(a) G_1

Figure: Graph sequence formed by successive dagger operations.



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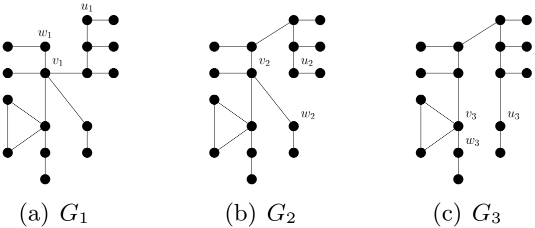

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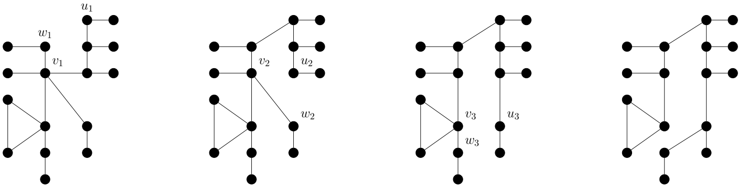

(a) G_1 (b) G_2 (c) G_3 (d) G_4

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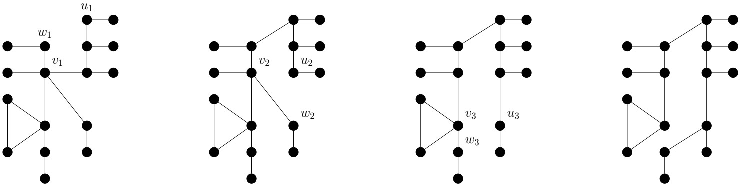

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Figure: Graph sequence formed by successive dagger operations.

Lemma (Beaton-C. 2020+): "This" works in general.

Even Order

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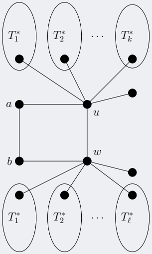
Proof sketch lower bound.: If $G \in \mathcal{KU}$ done. Else, $G \in \mathcal{S}_4$.



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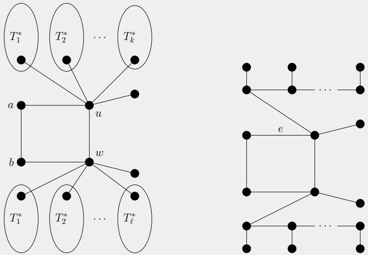

(a) G (b) F

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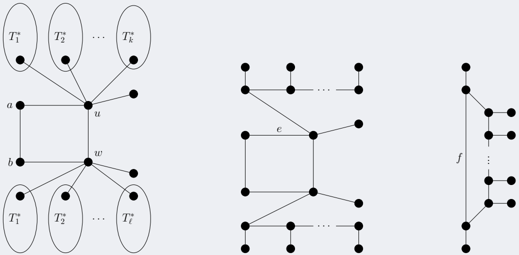

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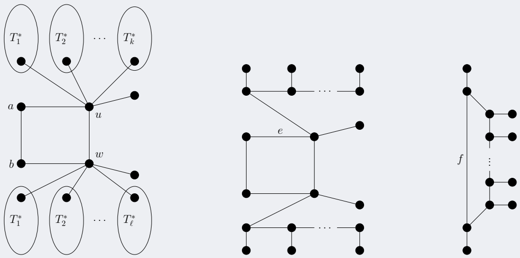

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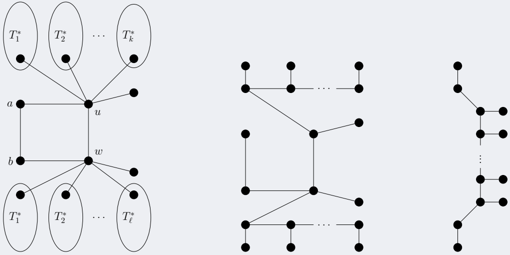

(a) G (b) $F-e$ (c) C_n^*-f

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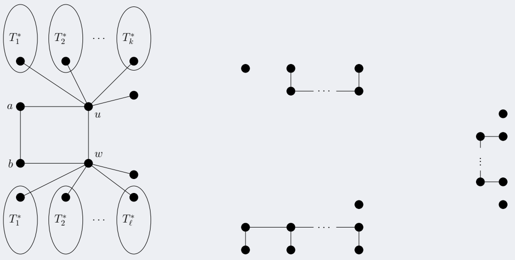

(a) G (b) $F - N[e]$ (c) $C_n^* - N[f]$

If $H - f \preceq G - e$ and $G - N[e] \preceq H - N[f]$, then $H \preceq G$.

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Theorem (Beaton-C. 2020+): If G is a connected well-covered unicyclic graph of order $2n$, then $C_n^* \preceq G \preceq U_n^*$.

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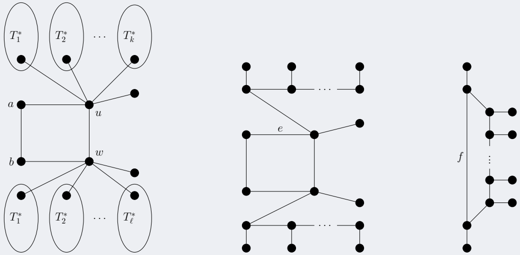
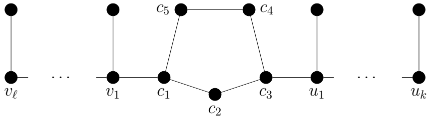


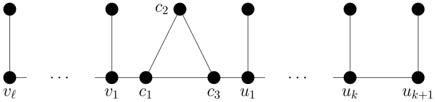
(a) G (b) F (c) C_n^*

So $G \succeq F \succeq C_n^*$. □

Odd Order



(a) $G(5, k, \ell)$



(b) $G(3, k+1, \ell)$

Figure: Graphs in the family $\mathcal{S}_P$.

Odd Order

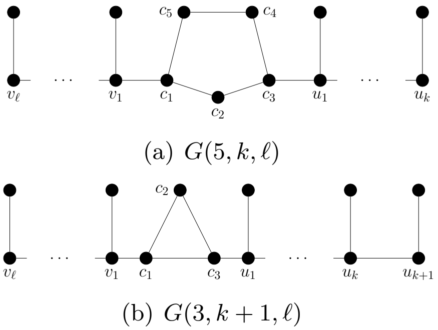


Figure: Graphs in the family $\mathcal{S}_P$.

Lemma (Beaton-C. 2020+) If $G, H \in \mathcal{S}_P$ and have the same order, then $I(G, x) = I(H, x)$.

Odd Order

Theorem (Beaton-C. 2020+) Let G be a connected well-covered unicyclic graph of odd order n and $H_n \in \mathcal{S}_P$ of order n. Then

i) $C_n \preceq G \preceq M_n$ if $n \leq 7$, and

ii) $H_n \preceq G \preceq M_n$ if $n \geq 9$.

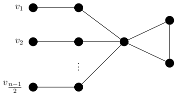


Figure: The graph M_n.

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i) $C_n \preceq G \preceq M_n$ if $n \leq 7$, and

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Corollary (Beaton-C. 2020+) Let G be a connected well-covered unicyclic graph of odd order n and $H_n \in \mathcal{S}_P$ of order n. Then

i) $\beta(C_n) \leq \beta(G) \leq \beta(M_n)$ if $n \leq 7$, and

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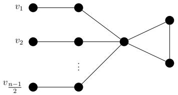

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Theorem (Beaton-C. 2020+): If G is a bipartite graph of order n, then $P_n \preceq G \preceq K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

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Conjecture: If G is a triangle-free graph of order n, then $P_n \preceq G \preceq K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Thank You!