The mean subtree order of a graph under edge addition

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(Joint work with Lucas Mol)

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December 4, 2020

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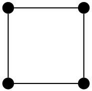

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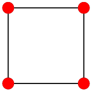

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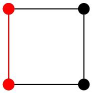

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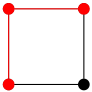

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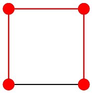

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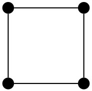

$$\mu(C_4) = \frac{4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + 4 \cdot 4}{4 + 4 + 4 + 4} = \frac{40}{16} = 2.5$$

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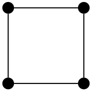
- The *subtree polynomial* of G is $S_G(x) = \sum_{T \in \mathcal{T}_G} x^{|V(T)|}$.



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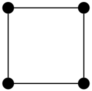

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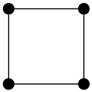

$S_{C_4}(x) = 4x + 4x^2 + 4x^3 + 4x^4$ $\qquad$ $\mu(C_4) = \frac{S'_{C_4}(1)}{S_{C_4}(1)} = \frac{40}{16}$.

Let $\mathcal{T}_G$ be the set of subtrees of G. Let $\mathcal{T}_{G,p}$ be the set of subtrees of G containing p (vertex or edge).

- The *subtree polynomial* of G is $S_G(x) = \sum_{T \in \mathcal{T}_G} x^{|V(T)|}$.
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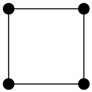

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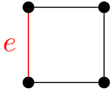

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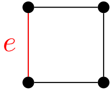
- The *local mean subtree order* of G at p, $\mu(G,p)$, is the average order of a subtree of G containing p.



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$$S_{C_4,e}(x) = x^2 + 2x^3 + 3x^4 \qquad \mu(C_4,e) = \frac{S'_{C_4,e}(1)}{S_{C_4,e}(1)} = \frac{20}{6}.$$

Note: For a graph G with p a vertex or edge, then

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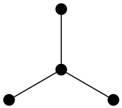

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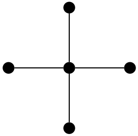

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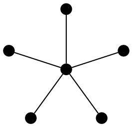

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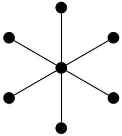

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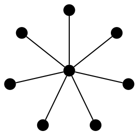

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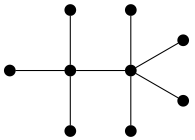

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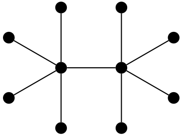

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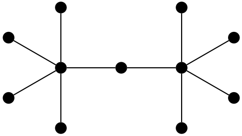

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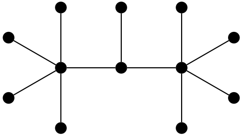

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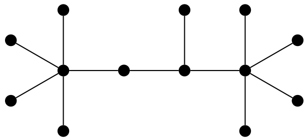

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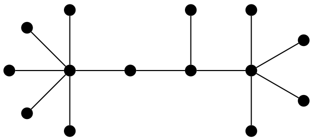

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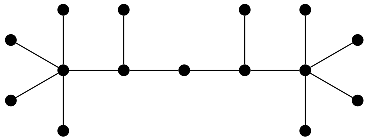

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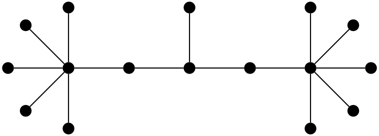

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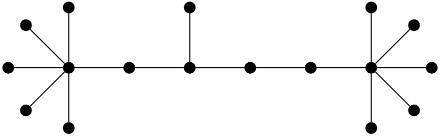

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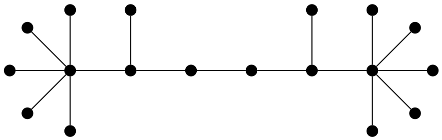

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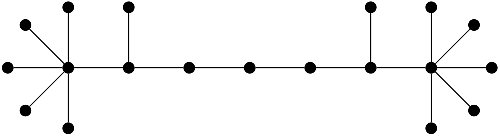

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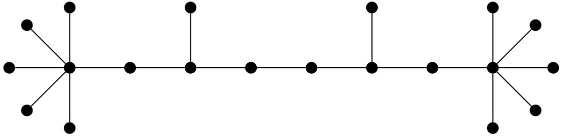

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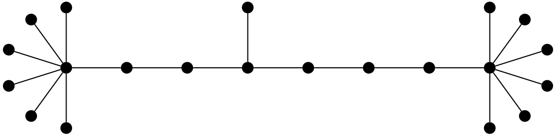

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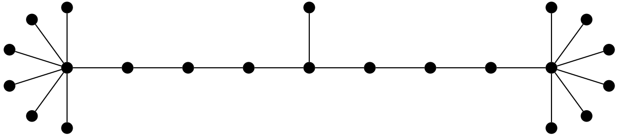

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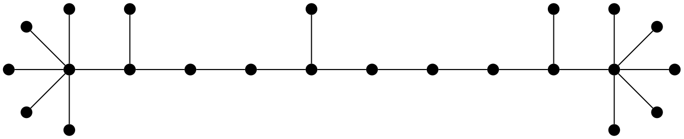

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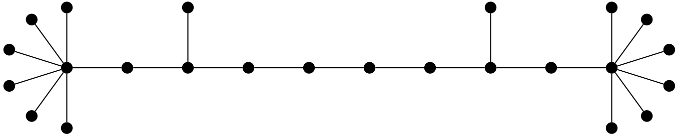

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In 2018, Chin, Gordon, MacPhee & Vincent extended the study of subtrees from trees to graphs by considering:

- The sutbree polynomial, $S_G(x)$ of graphs.
- The shape of the coefficient sequence of $S_G(x)$.
- The probability that a randomly chosen tree is spanning.
- The mean subtree order of a graph G, $\mu(G)$.

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Conjecture (Chin et al. 2018): Suppose that G is a connected multigraph, and that H is obtained from G by adding an edge between two distinct vertices of G. Then $\mu(G) < \mu(H)$.

In 2018, Chin, Gordon, MacPhee & Vincent extended the study of subtrees from trees to graphs by considering:

- The sutbree polynomial, $S_G(x)$ of graphs.
- The shape of the coefficient sequence of $S_G(x)$.
- The probability that a randomly chosen tree is spanning.
- The mean subtree order of a graph G, $\mu(G)$.

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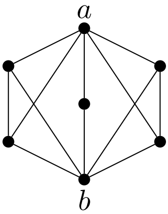

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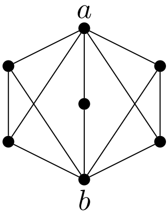

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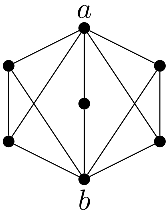

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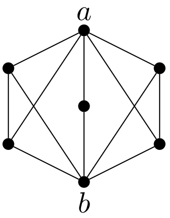

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- 347 counterexamples of order 8!

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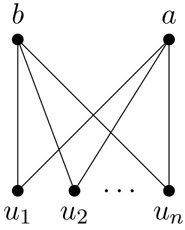

Figure: $K_{2,n}$

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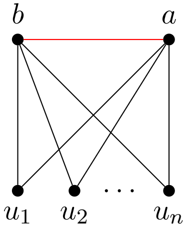

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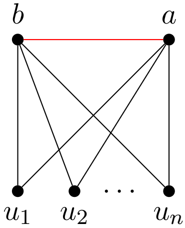

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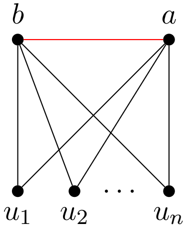

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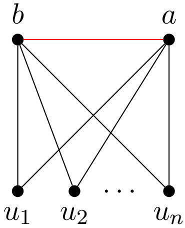

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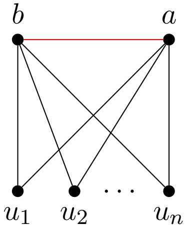

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Maybe adding an edge cannot decrease $\mu(G)$ by too much?

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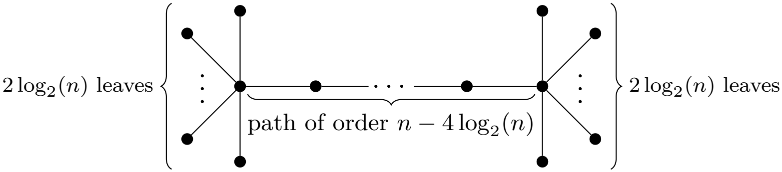

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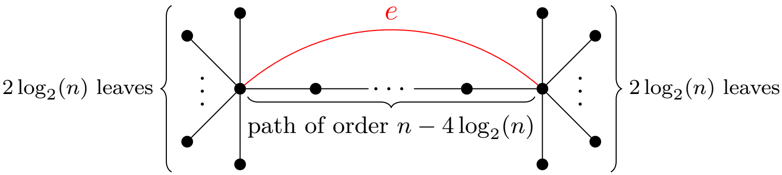

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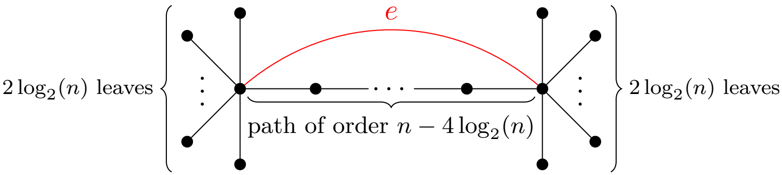

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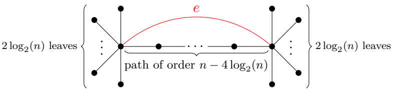
Show $\lim_{n\to\infty} \mathrm{Den}(T_n) - \mathrm{Den}(G_n) = \frac{1}{3}$.



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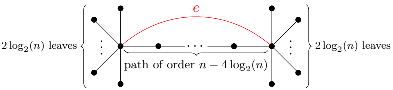

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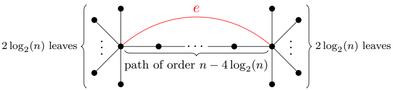

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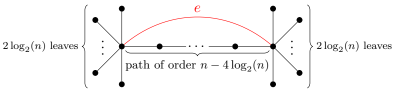

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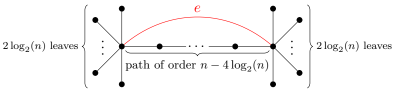

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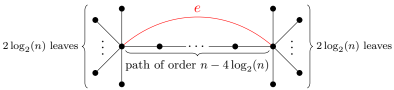

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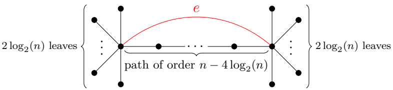

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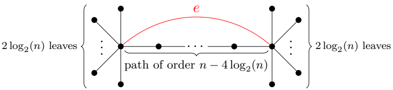

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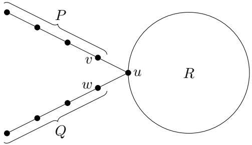

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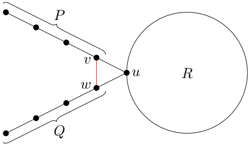

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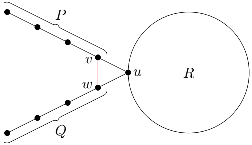

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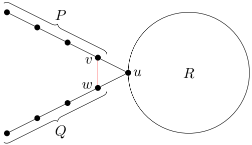

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Local/Global Mean Inequality (Jamison 1983): If T is a tree, then for all $u \in V(T)$, $\mu(T, v) \geq \mu(T)$ with equality if and only if $T = K_1$.

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Proposition (C.-Mol 2020): Let G be a multigraph of order at least 2. Then there is a multigraph H, obtained from G by adding a new edge between a pair of distinct vertices of G, such that $\mu(H) > \mu(G)$.

Open Problems

Conjecture: Suppose G is a connected graph which is not complete. Then there is a graph H, obtained from G by joining two distinct, nonadjacent vertices, such that $\mu(H) > \mu(G)$.

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Problem: Suppose that a graph H is obtained from a connected graph G by adding an edge between two nonadjacent vertices of G. Determine sharp bounds on $\mathrm{Den}(H) - \mathrm{Den}(G)$.

Thank You!

Figure: A mean subtree.