k-critical H-free graphs: The Final Frontier - Part 1

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(Joint work with Tala Abuadas, Iain Beaton, Chính Hoàng, and Joe Sawada)

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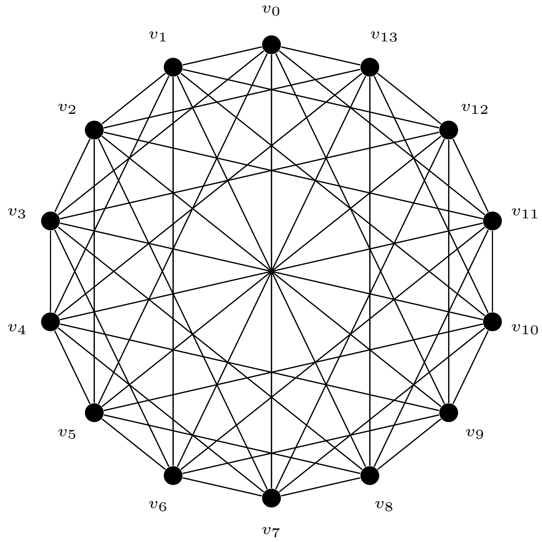
Warm-up 1: Is the following graph bipartite?



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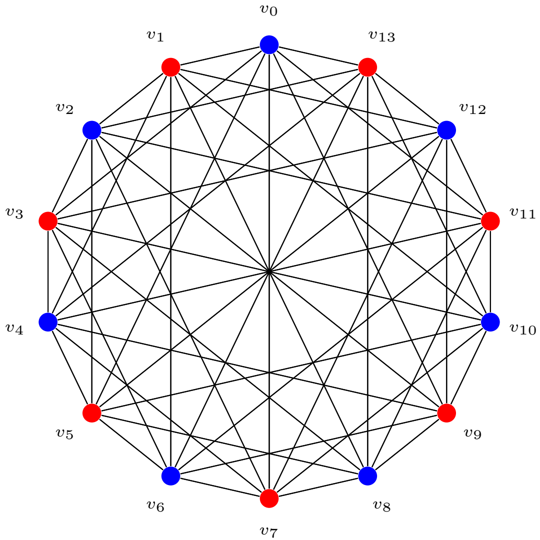

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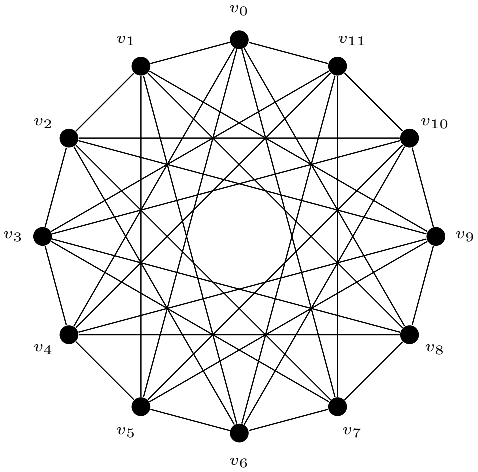

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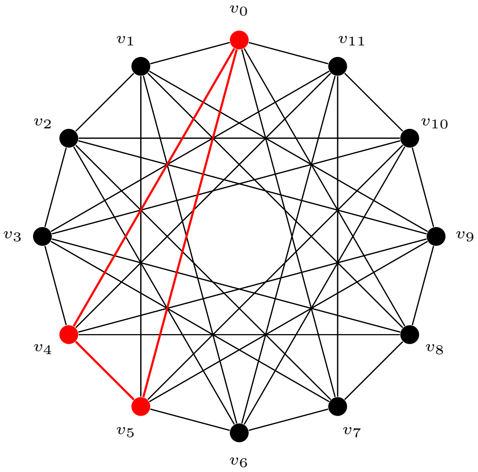

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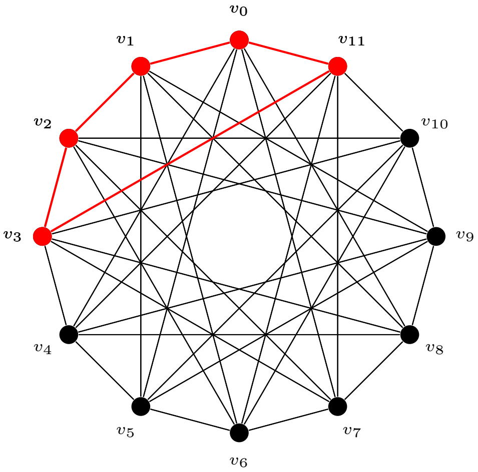

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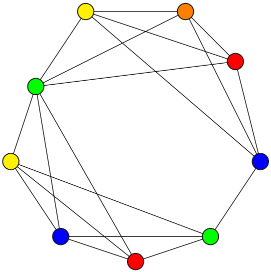

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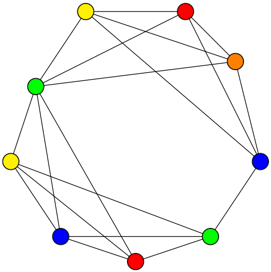

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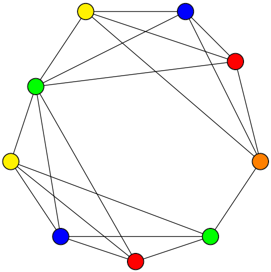

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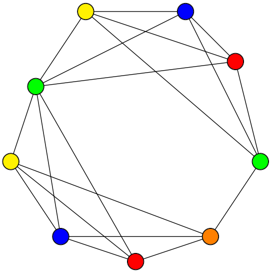

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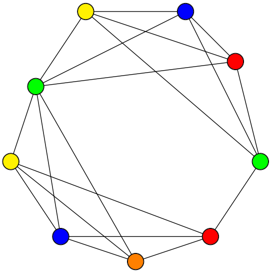

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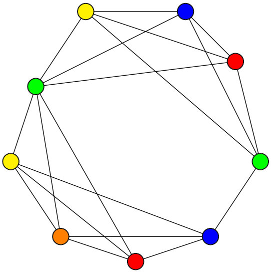

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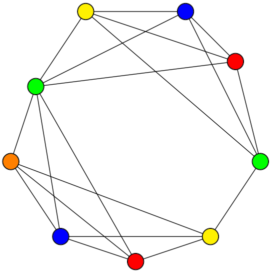

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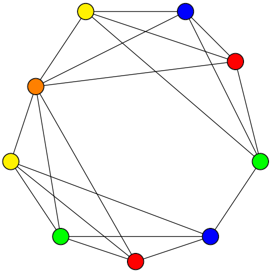

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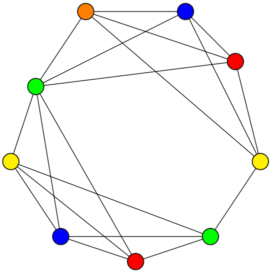

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($\Leftarrow$) If G contains a $(k+1)$-critical induced subgraph H, but G is k-colourable, then:

- We can use a k-colouring of G restricted to $V(H)$ to k-colour H, a contradiction!

Definition: A certifying algorithm is an algorithm that produces with each output a certificate that the particular output has not been compromised by a bug. 

Consider an algorithm to solve IsBipartite.

- If an input graph is bipartite, then the "yes"-certificate is a bipartition.
- If an input graph is non-bipartite, then the "no"-certificate is an induced odd cycle.

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- Bottom line: IsBipartite can be efficiently certified.

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- If an input graph is k-colourable, then the "yes"-certificate is a k-colouring.
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- Bottom line: IS-k-COLOURABLE can be efficiently "yes"-certified, but likely not efficiently "no"-certified :-(

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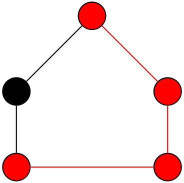

Figure: The graph C_5 is P_5-free, but not P_4-free

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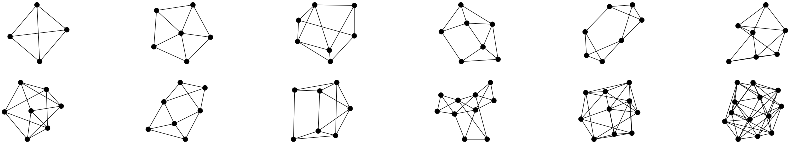

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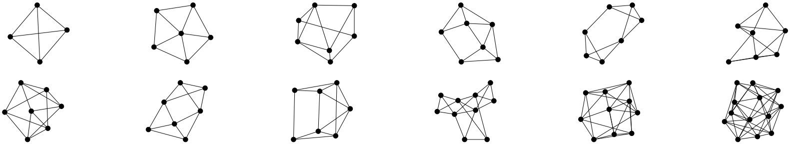

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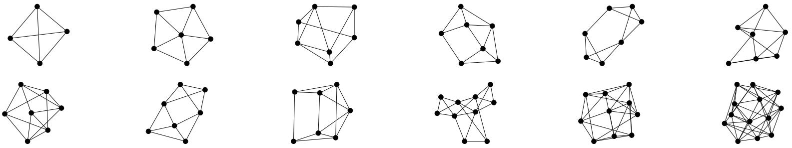

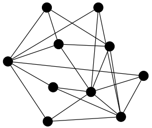

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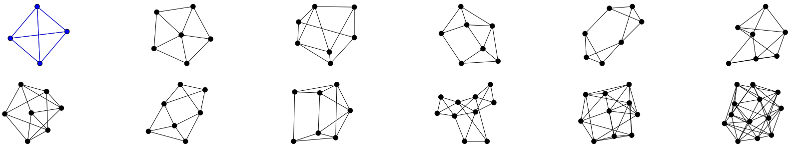

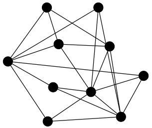

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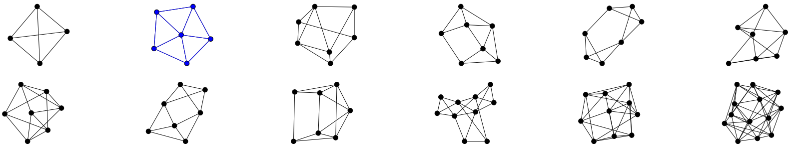

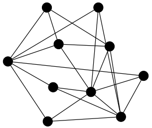

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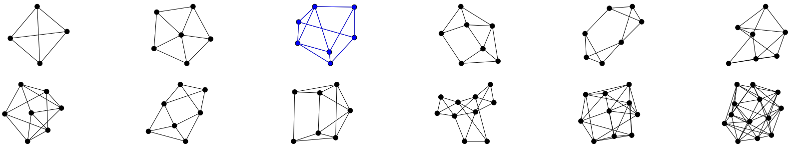

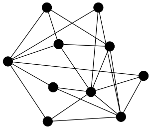

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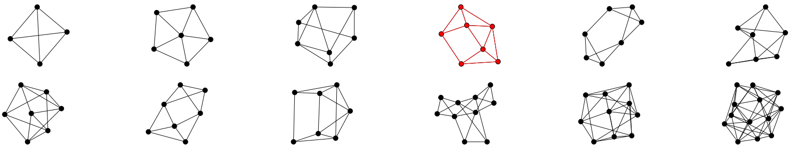

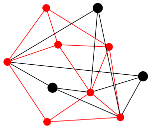

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- ℓP_1 Follows from Ramsey's Theorem!
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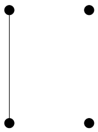
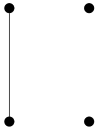


Figure: $P_2 + 2P_1$, smallest open case of "Critical" Question in 2020.



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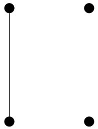

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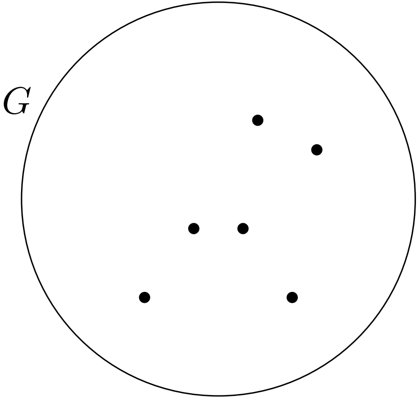

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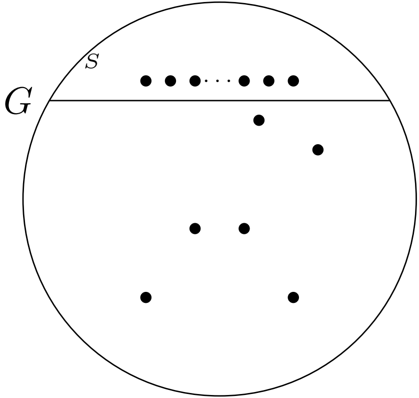

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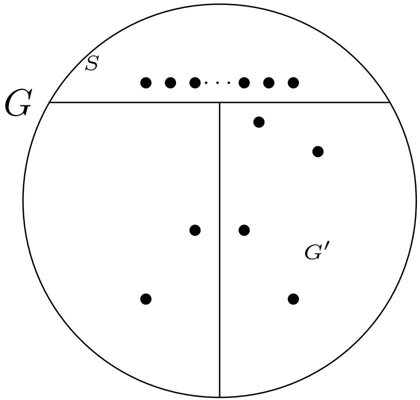

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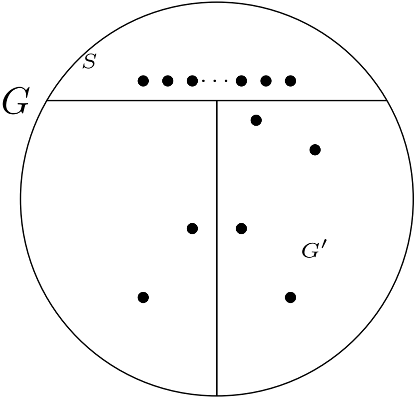

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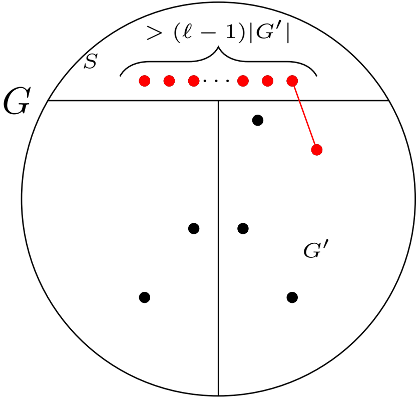

- G:k-critical $(P_2 + \ell P_1)$-free
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- G':$(k-1)$-critical
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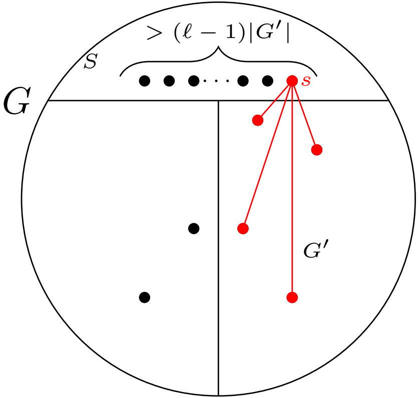

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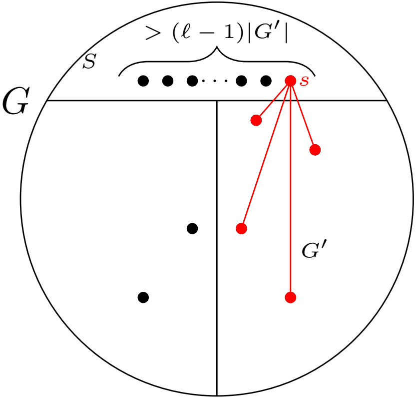

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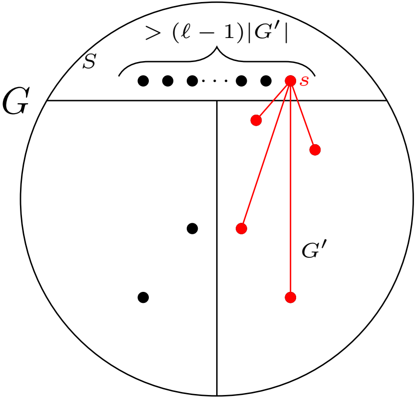

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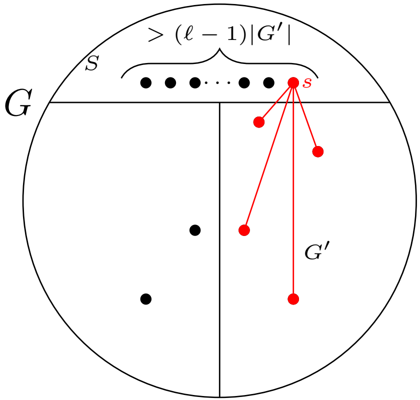

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Contradiction!

"Critical" Question only open for $H = P_3 + \ell P_1$ and $P_4 + \ell P_1$ for $\ell > 0$.

Theorem (Abuadas-C.-Hoàng-Sawada 2024): There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.

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"Critical" Question: For which H are there only finitely many k-critical H-free graphs for all k?

Thus, for the "Critical" Question, H must be one of:

- ℓP_1 (Follows from Ramsey's Theorem!)
- $P_2 + \ell P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + \ell P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- $P_4 + \ell P_1$

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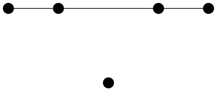
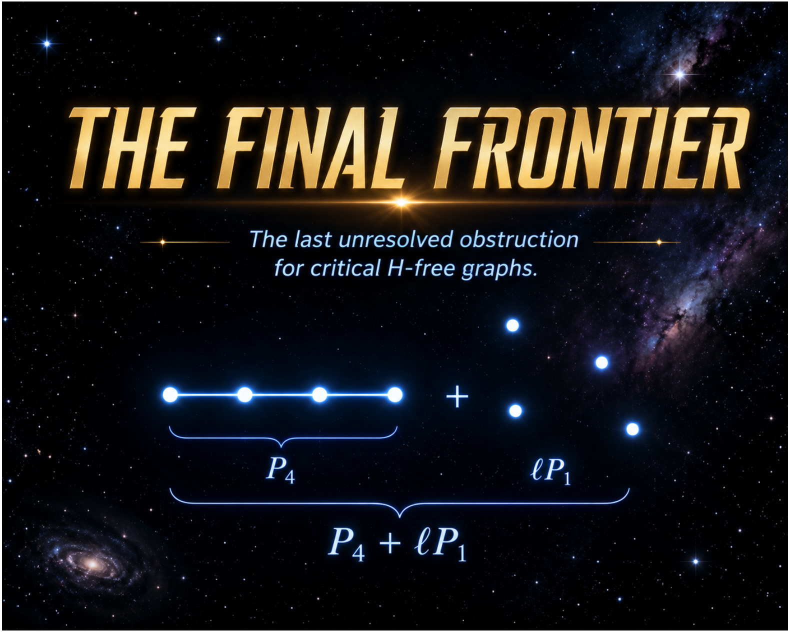


Figure: $P_4 + P_1$, the smallest open H for the "Critical" Question.


THE FINAL FRONTIER
The last unresolved obstruction
for critical H-free graphs.
P_4
ℓP_1
$P_4 + \ell P_1$

To be continued in part 2 by...

Thank You!

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