An infinite family of k-critical $(2P_3, K_{k-1})$-free graphs for each $k \geq 5$

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(Joint work with Iain Beaton)

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Definition: For fixed k, the k-Coloring decision problem is to determine if a given graph is k-colorable.

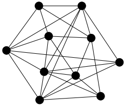

Figure: Decide 3-Coloring for this graph.

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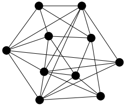

Figure: Decide 3-COLORING for this graph.

- k-COLORING is NP-complete for all $k \geq 3$ (Karp 1972).

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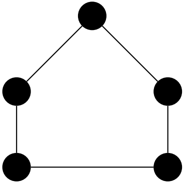

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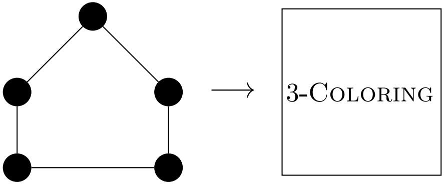

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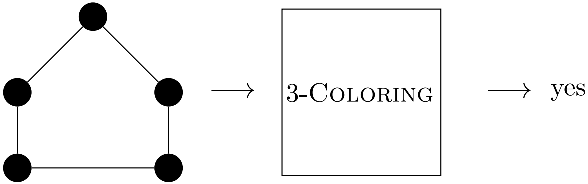

Theorem (Hoàng et al. 2010) k-Coloring P_5-free graphs can be solved in polynomial-time for all k and the algorithm gives a valid k-coloring if one exists.

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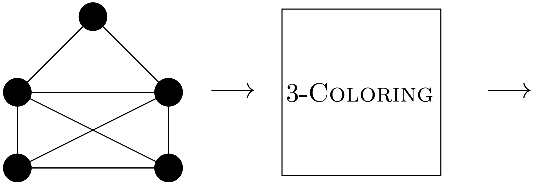

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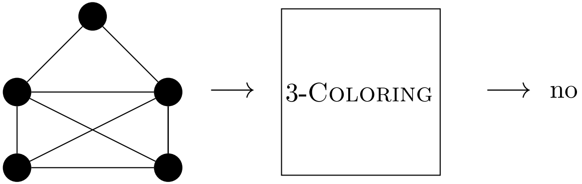

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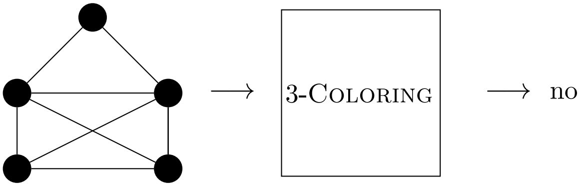

- A k-coloring is a certificate to verify a "yes".
- How can we verify a "no"?

- A graph G is k-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.

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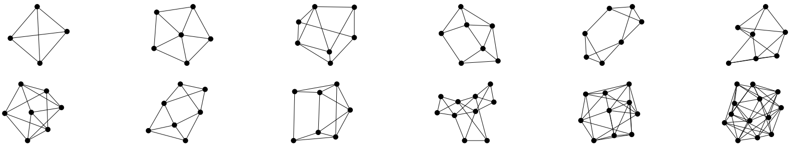
Issue: For $k \geq 4$ there is no known polynomial-time recognition algorithm for k-critical graphs.

Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.



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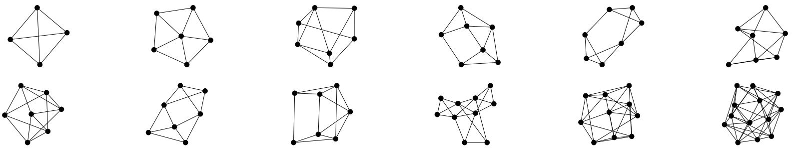

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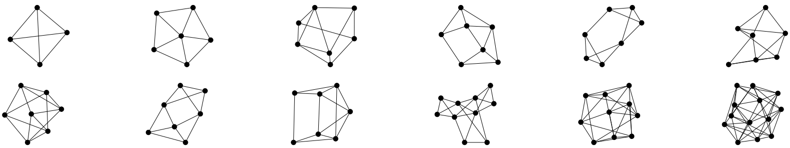

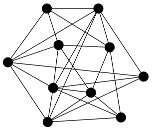

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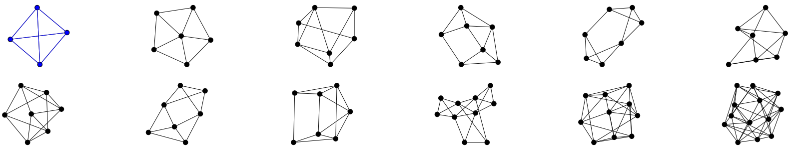

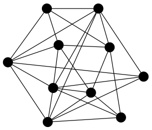

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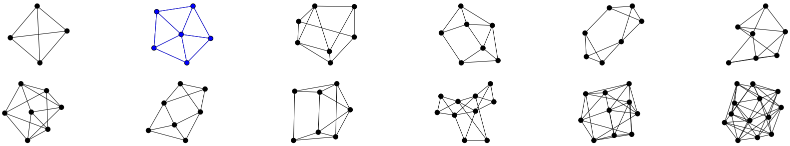

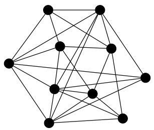

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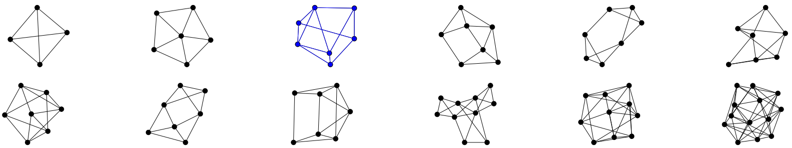

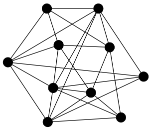

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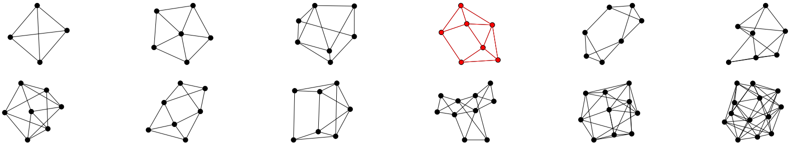

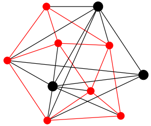

Question 1: For which H are there only finitely many k-critical H-free graphs for all k? (And therefore polynomial-time certifying $(k-1)$-Coloring algorithms for all k)

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Thus, if there are only finitely many k-critical H-free graphs for any $k \geq 3$, then H must be a linear forest.

Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are only finitely many 4-critical H-free graphs if and only if H is an induced subgraph of:

- P_6 (),
- $P_4 + \ell P_1$ for some $\ell \geq 0$ (), or
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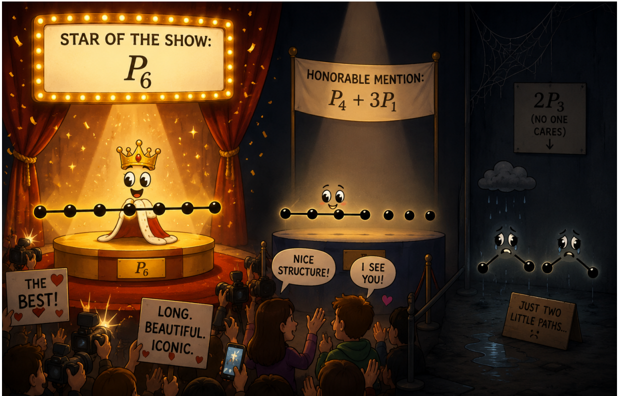

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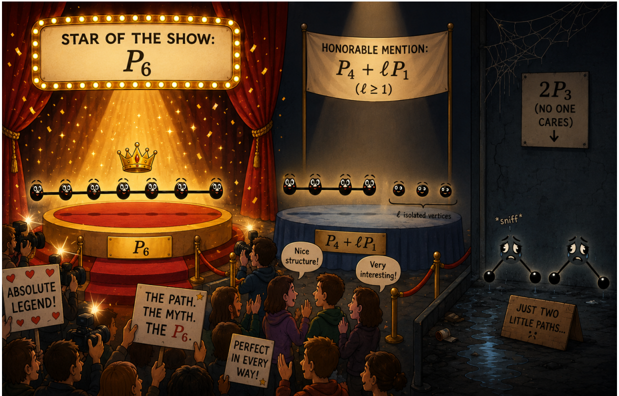

Theorem (Huang-Li-Shi 2019): There are only finitely many 5-critical (P_6, banner)-free graphs.

Theorem (Ju-Jooken-Goedgebeur-Huang 2026): There are only finitely many 5-critical (P_6, bull)-free graphs.

But, the even more restrictive P_5-free graphs have received even more attention...

Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): For H of order 4 and $k \geq 5$, there are infinitely many k-critical (P_5, H)-free graphs if and only if H is $2P_2$ or $K_3 + P_1$.

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Also finite if H is any of the graphs below:

- $\overline{K_5}$
- $\overline{P_5}$
- banner
- $\overline{P_3 + P_2}$ or gem
- $K_{2,3}$ or $K_{1,4}$
- dart
- $P_2 + 3P_1$
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$
- $P_3 + 2P_1$
- W_4
- chair or cricket
- bull

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Also finite if H is any of the graphs below:

- $\overline{K_5}$ (Ramsey 1928)
- banner (Brause-Geißer-Schiermeyer 2022)
- $K_{2,3}$ or $K_{1,4}$ (Kamiński-Pstrucha 2019)
- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- chair or cricket (Jooken 2026+)
- $\overline{P_5}$ (Dhaliwal-Hamel-Hoàng-Maffray-McConnel-Panait 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)
- dart (Xia-Jooken-Goedgebeur-Huang 2023)
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$ (Xia-Jooken-Goedgebeur-Huang 2024)
- W_4 (Xia-Jooken-Goedgebeur-Beaton-C.-Huang 2025)
- bull (Belavadi-Hoàng 2026+)

There have been 3 papers directly on k-critical $(P_4 + \ell P_1)$-free:

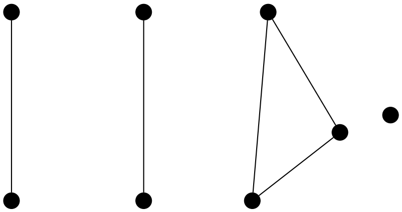
- Abuadas-C.-Hoàng-Sawada 2024
- Beaton-C. 2025
- Beaton-C. 2026.

Figure: This fun guy will tell you about this in the next talk!

Theorem (Hoàng et al. 2015): There are infinitely many k-critical $(2P_2, K_3 + P_1)$-free graphs for all $k \geq 5$.

Theorem (C.-Hoàng 2024): There are infinitely many k-critical $(2P_2, K_3 + P_1, C_5)$-free graphs for all $k \geq 6$.

Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are infinitely many k-critical P_7-free for all $k \geq 4$.



(a) $2P_2$ (b) $K_3 + P_1$

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Theorem (Chudnovsky-Hajebi-Spirkl 2024): For all $k, r \geq 1$, List-k-Coloring rP_3-free graphs is polynomial-time solvable.

Theorem (Beaton-C. 2026++): There are infinitely many k-critical $(2P_3, 3P_2, K_{k-1})$-free graphs for all $k \geq 5$.

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Let $G'(p,k)$ be a graph on vertex set $\{v_0, v_1, ..., v_{(k-1)p}\}$ where the neighbourhood of vertex v_i is the union of the following three sets

- $\{v_{i+j} : j = 1, 2, \dots, k-3\}$
- $\{v_{i-j} : j = 1, 2, \dots, k-3\}$
- $\{v_{i+(k-1)j+m} : m = 2, 3, ...k-2 \text{ and } j = 1, ..., p-2\}$.

(with each integer taken modulo $(k-1)p + 1$).

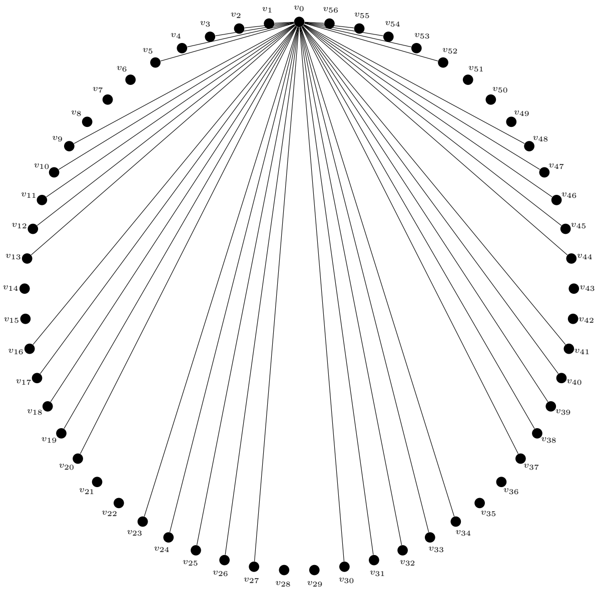


Figure: $N(v_0)$ in the 8-critical graph $G'(8,8)$.

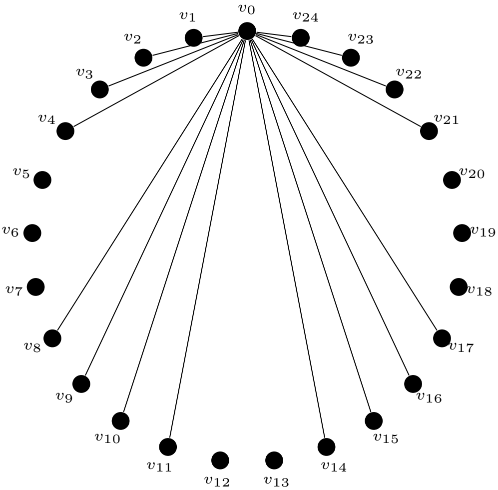
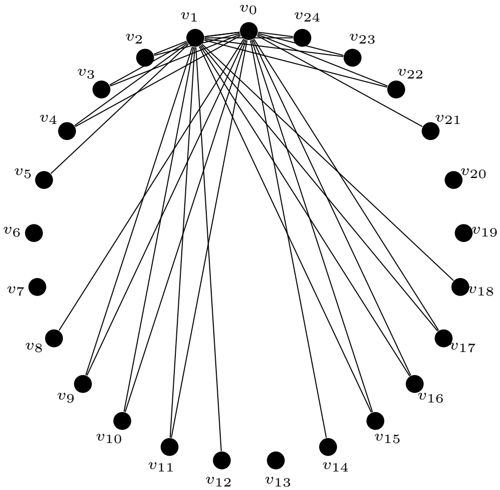


Figure: Constructing and colouring the 7-critical graph $G'(4, 7)$.



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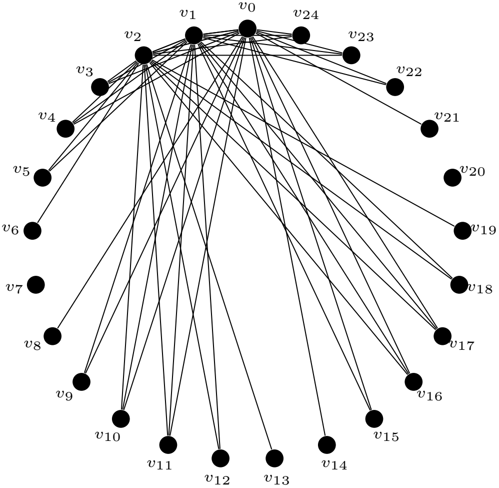

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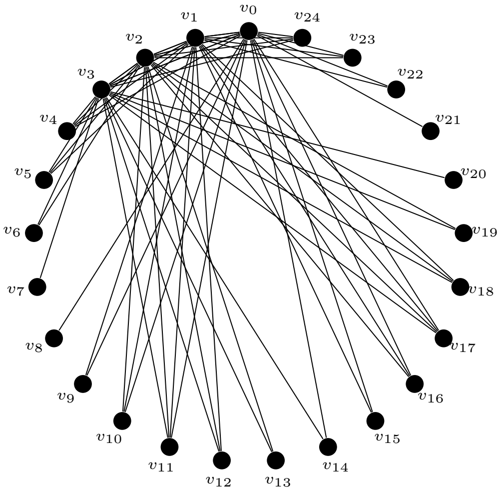

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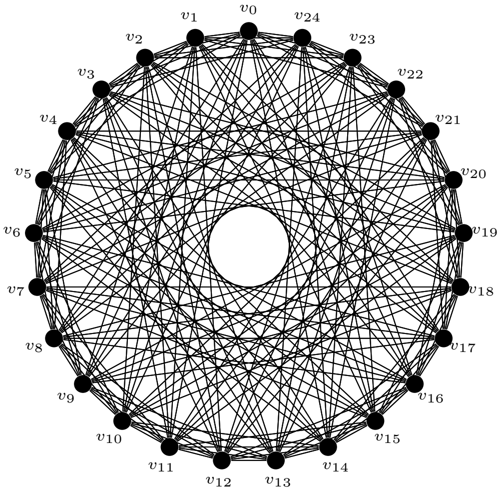

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It was the $2P_3$-freeness that caused us the most trouble!

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Thank You!

Atlantic Association for Research in the Mathematical Sciences

NSERC CRSNG

Figure: Thanks to NSERC for research support and AARMS for session support!