# Efficient Graph Colouring with Certificates:

## Exploiting Mathematical Structure to Design Better Algorithms

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**UPEI**

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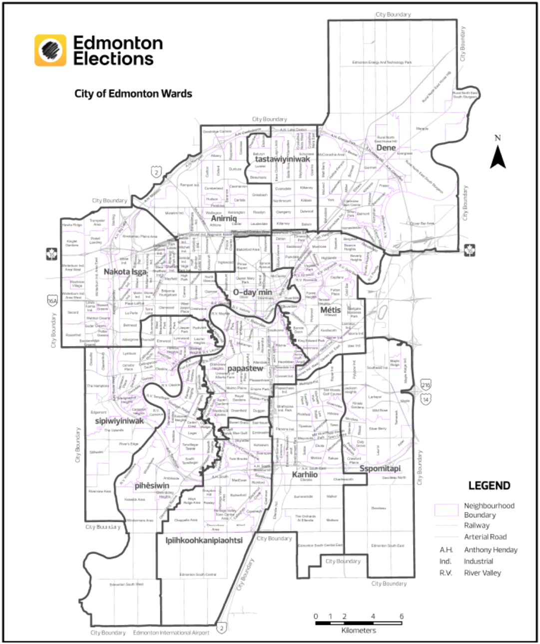
Adadia University Math Seminar

July 14, 2026

Four Colour Problem (1852): Are four colours sufficient to colour *any* map so that bordering regions receive different colors?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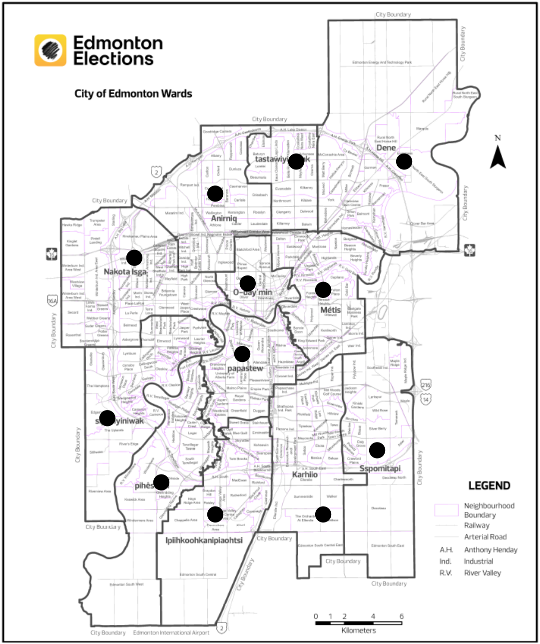
Kilometers
Edmonton International Airport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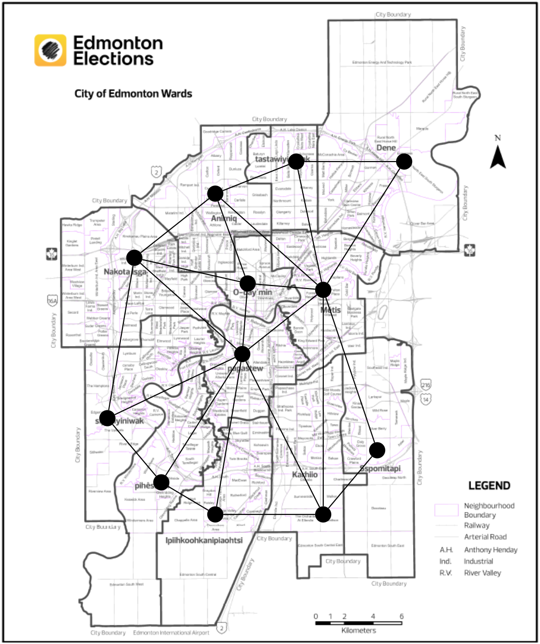
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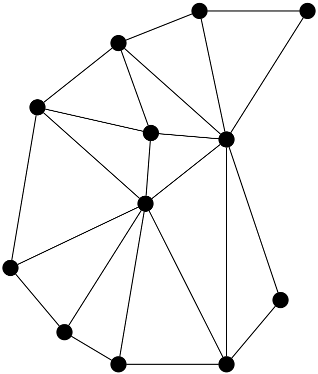

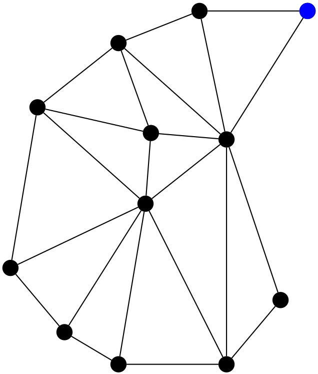

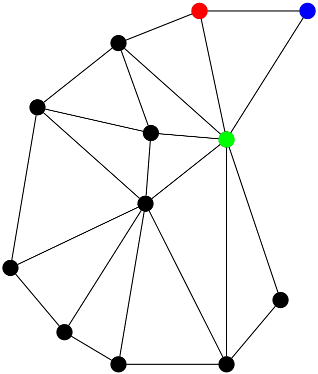

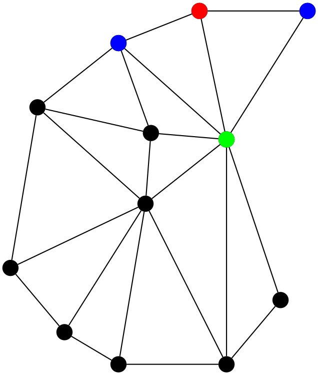

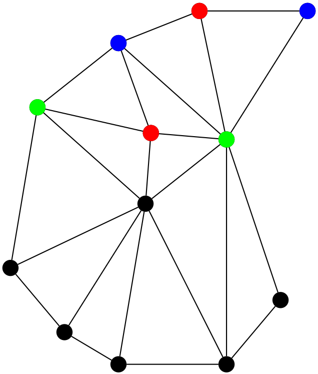

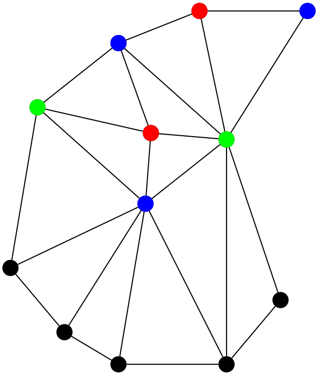

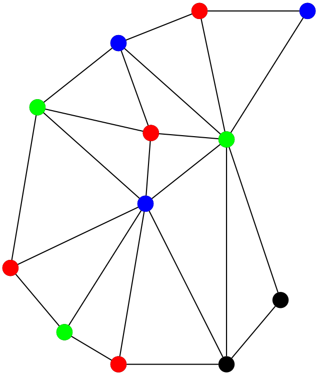

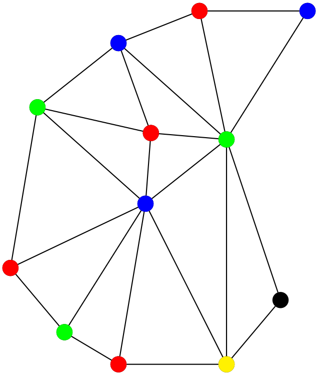

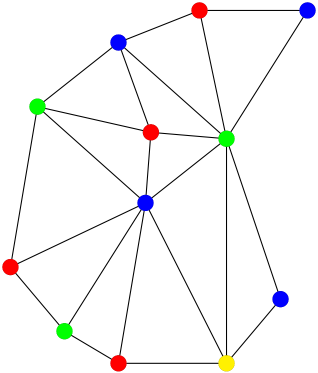

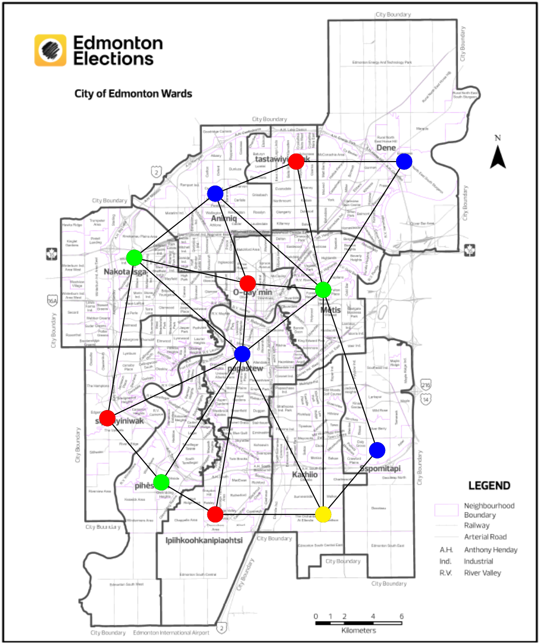
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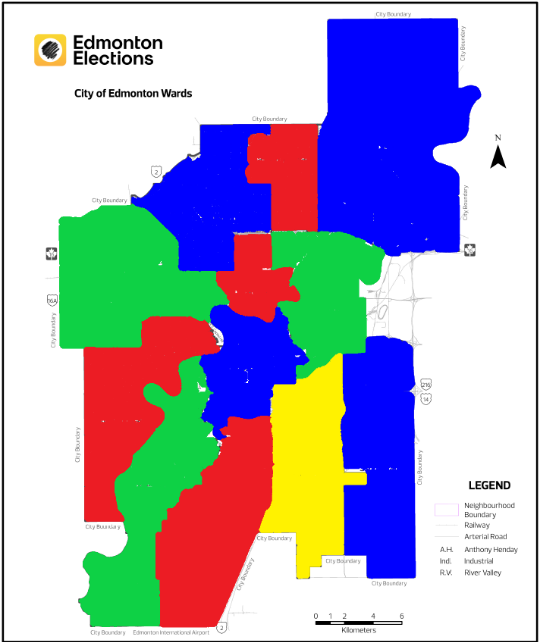
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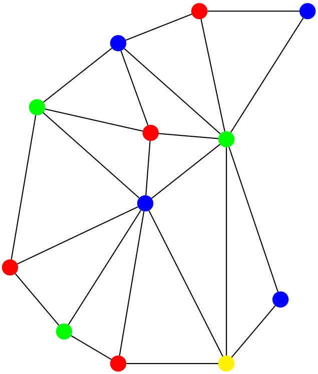
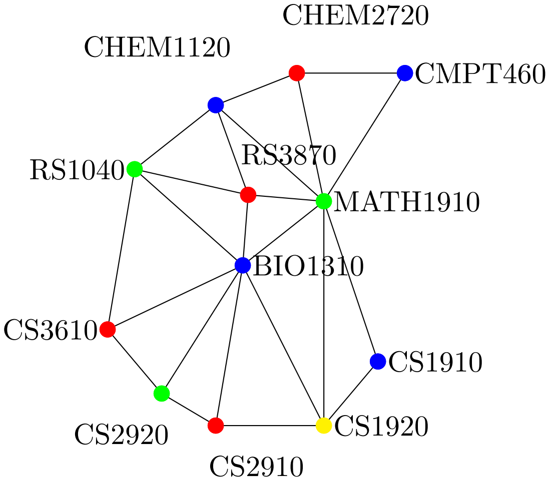
Four Colour Problem (1852): Are four colours sufficient to colour *any* map so that bordering regions receive different colors?

Four Colour Theorem (Appel-Haken 1976): Four colours suffice to colour *any* map so that bordering regions receive different colors.

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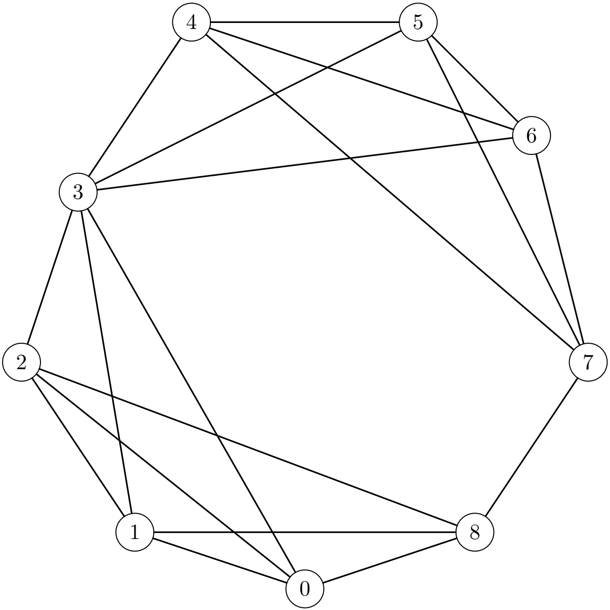
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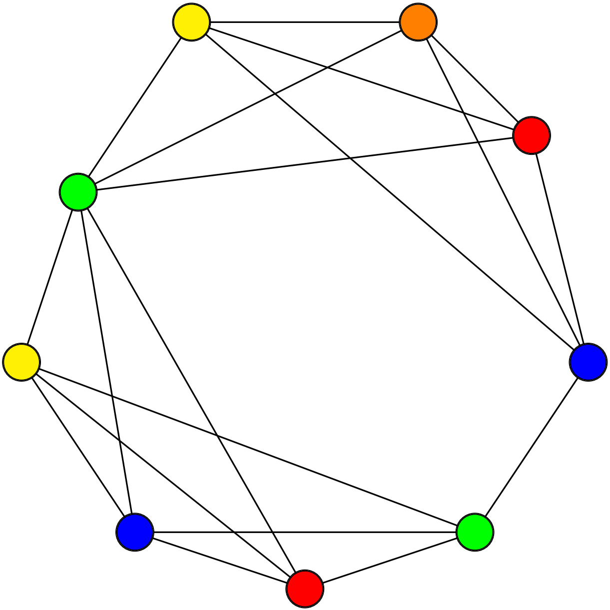

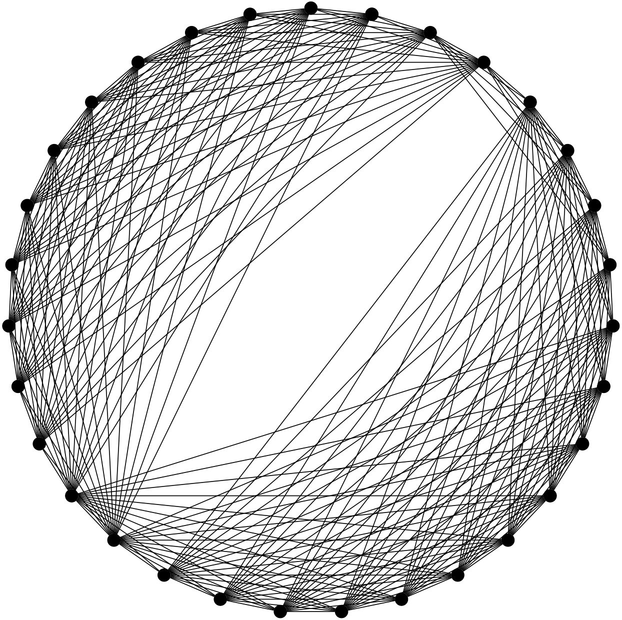

**Fact:** For any $k$, there is a graph requiring at least $k$ colours.

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- Did we solve the 4-COLOURING problem for the handout graph?
- How hard is it to decide the 4-COLOURING problem in general?


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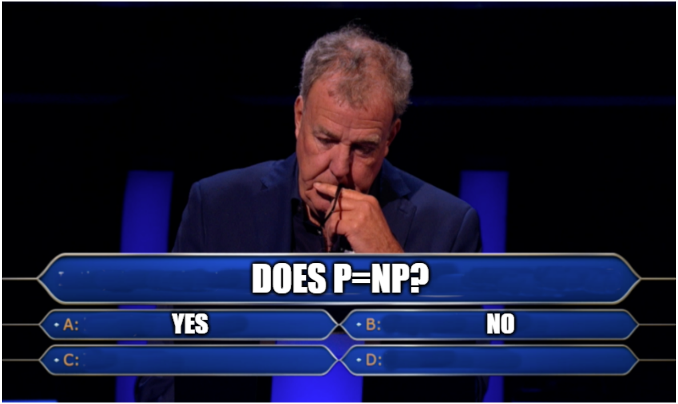
DOES P=NP?
A: YES
B: NO
C:
D:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | | | 7 | | | | |
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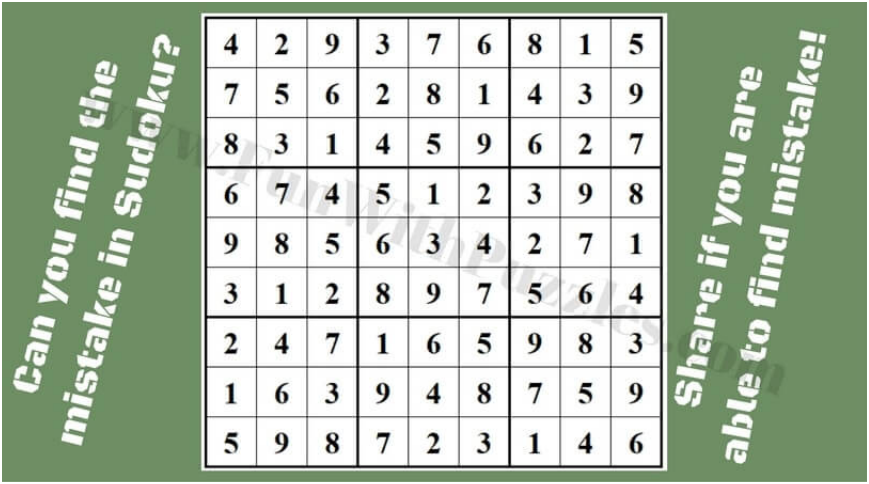
Can you find the mistake in Sudoku?
4 2 9 3 7 6 8 1 5
7 5 6 2 8 1 4 3 9
8 3 1 4 5 9 6 2 7
6 7 4 5 1 2 3 9 8
9 8 5 6 3 4 2 7 1
3 1 2 8 9 7 5 6 4
2 4 7 1 6 5 9 8 3
1 6 3 9 4 8 7 5 9
5 9 8 7 2 3 1 4 6
Share if you are able to find mistake!
www.FunWithPuzzles.com

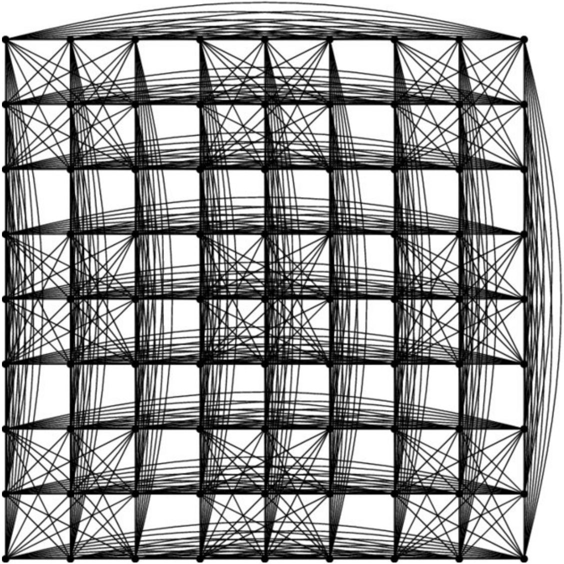

Figure: The $9 \times 9$ Sudoku Graph (image from *Colouring sparse graphs*, François Pirot, PhD Thesis, 2019)

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**Answer:** There are structural differences between map graphs and general graphs that can be exploited to develop efficient $k$-COLOURING algorithms. (FYI, these graphs are called *planar*)

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I am interested in developing polynomial-time $k$-Colouring algorithms for families of graphs defined by forbidden substructures.

**Theorem** (Hoàng et al. 2010) $k$-COLOURING on $P_5$-free graphs can be solved in polynomial-time for all $k$ and the algorithm gives a valid $k$-colouring if one exists.

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$P_5$ is of particular interest because:

1. It is the largest connected induced subgraph that can be forbidden for which there exits polynomial-time $k$-Colouring algorithms for all $k$. (Huang 2016)
2. Many graphs in nature are $P_5$-free or close to $P_5$-free (i.e., think of the Six Degrees of Separation Theory).

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Question: Can we make the above algorithms better in any way?

**Theorem** (Hoàng et al. 2010) $k$-Colouring on $P_5$-free graphs can be solved in polynomial-time for all $k$ and the algorithm gives a valid $k$-colouring if one exists. What if the answer is no?
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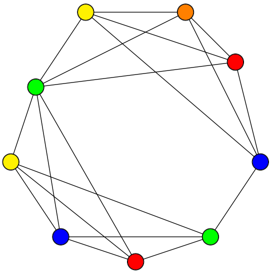
To make these algorithms certifying, we need some easy-to-verify minimal substructres that will prevent $P_5$-free graphs from being $k$-colourable.

**Def:** A graph $G$ is $k$-critical if $G$ is not requires $k$ colors, but deleting any vertex results in a $(k-1)$-colorable graph.



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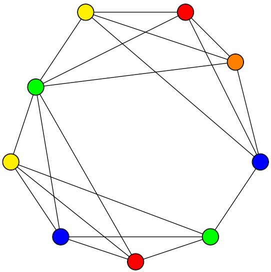

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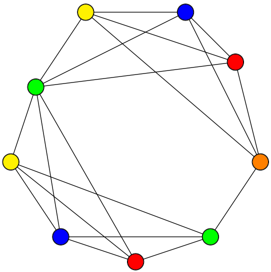

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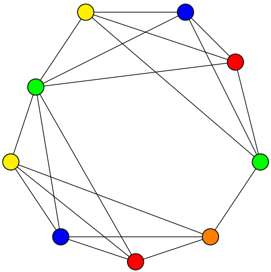

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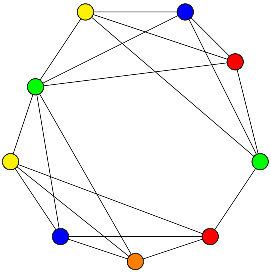

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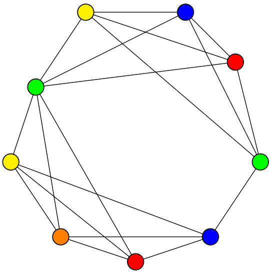

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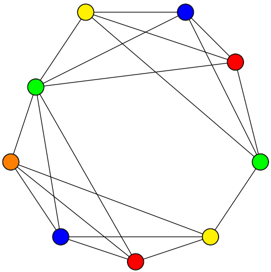

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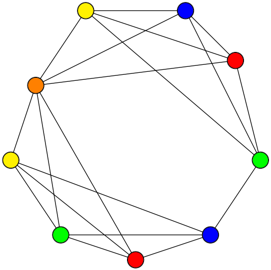

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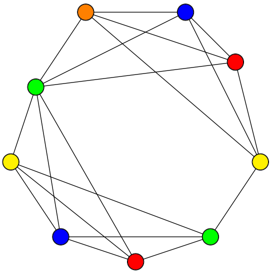

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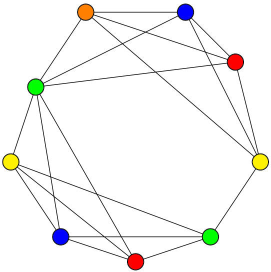

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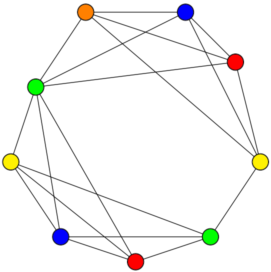

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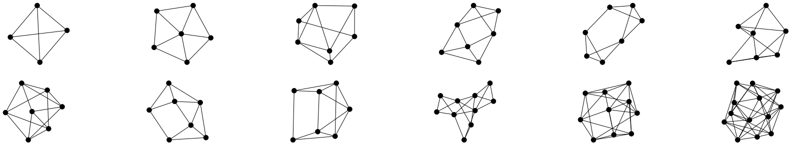
- Idea: Let's use the list of all $k$-critical graphs to make an efficient $(k-1)$-Coloruing algorithm!

**Theorem (Bruce et al. 2009, Maffray-Morel 2012)** All 4-critical $P_5$-free graphs are given below. Proof uses exhaustive computer search



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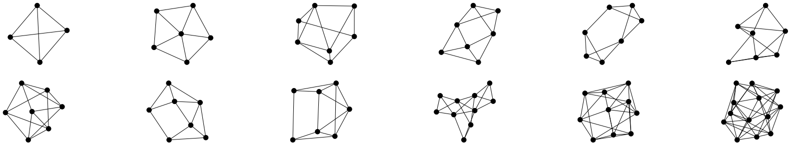

Finite number of $k$-critical graphs implies polynomial-time $(k-1)$-Coloring algorithm with "no"-certificates.

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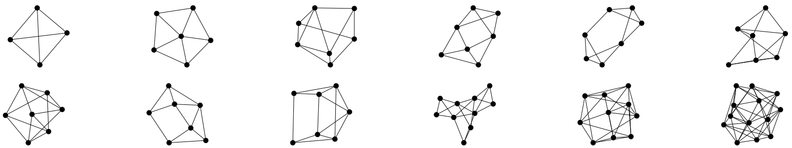

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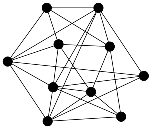

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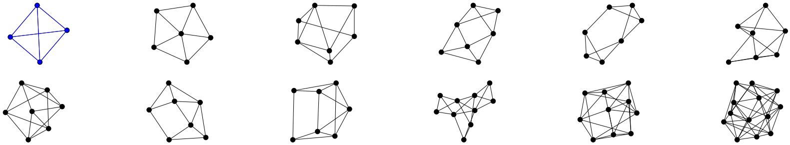

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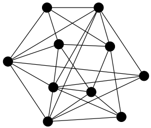

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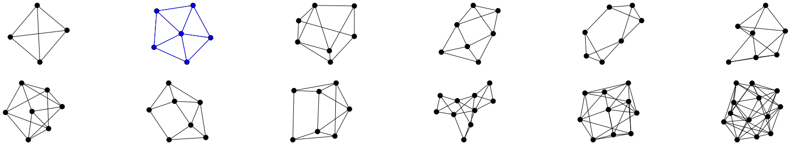

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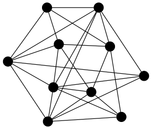

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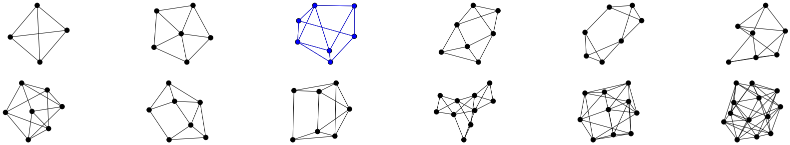

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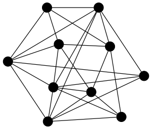

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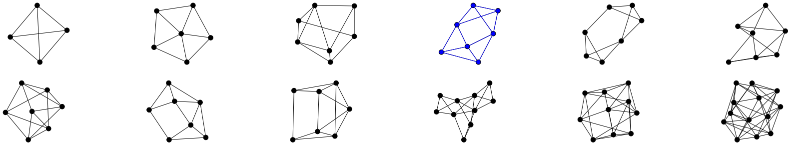
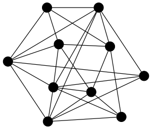


Finite number of $k$-critical graphs implies polynomial-time $(k-1)$-COLORING algorithm with "no"-certificates.



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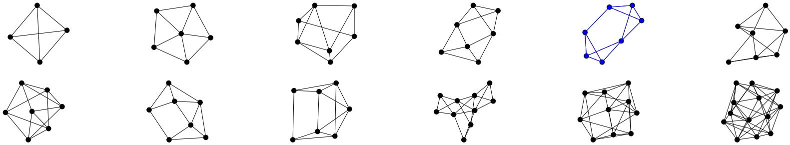

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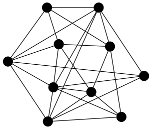

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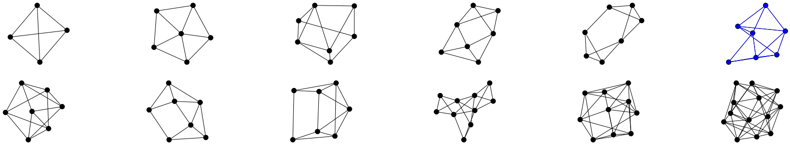

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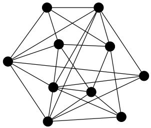

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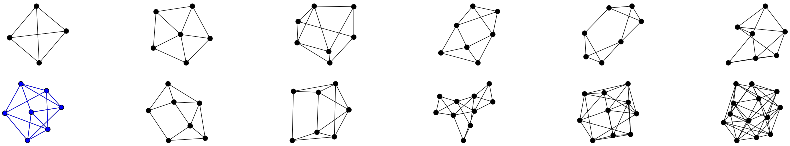

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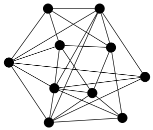

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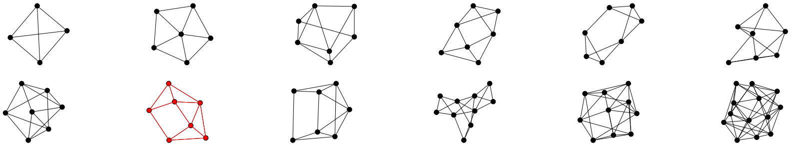

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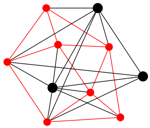

The simplicity and ability to easily debug implementations of the previous algorithm shouldn't be missed!

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Four-coloring $P_6$-free graphs.
I. Extending an excellent precoloring

Maria Chudnovsky*
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Sophie Spirkl
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November 16, 2021

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II. Finding an excellent precoloring

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- Since these algorithms are still desirable, I'm interested in subclasses of $P_5$-free graphs that admit fully certifying $k$-Colouring algorithms for all $k$.
- In particular, subclasses defined by forbidding an additional induced subgraph (i.e., $(P_5, H)$-free graphs for various graphs $H$).

Question: For which graphs $H$ are there only finitely many $k$-critical $(P_5, H)$-free graphs **for all** $k$?

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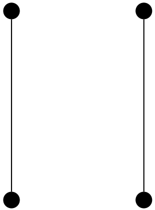
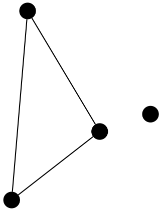
**Theorem** (K. Cameron-Goedgebeur-Huang-Shi 2021): For $H$ of order 4 and $k \geq 5$, there are only finitely many $k$-critical $(P_5, H)$-free graphs if and only if $H$ is **not** $2P_2$ or $K_3 + P_1$.



(a) $2P_2$



(b) $K_3 + P_1$

Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs $H$ of **order 5** are there only finitely many $k$-critical $(P_5, H)$-free graphs for all $k$?

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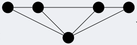
True if $H$ is any of the graphs below:

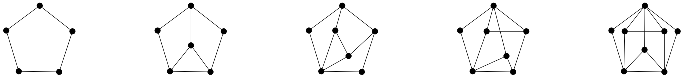
- $\overline{K_5}$
- banner
- $K_{2,3}$ or $K_{1,4}$
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- chair or cricket
- $\overline{P_5}$
- $\overline{P_3 + P_2}$ or gem
- dart
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$
- $W_4$
- bull

**Open Problem** (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs $H$ of **order 5** are there only finitely many $k$-critical $(P_5, H)$-free graphs for all $k$?

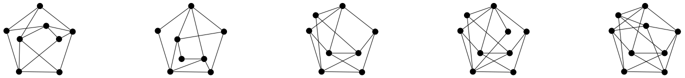
True if $H$ is any of the graphs below:

- $\overline{K_5}$ (Ramsey 1928)
- banner (Brause-Geißer-Schiermeyer 2022)
- $K_{2,3}$ or $K_{1,4}$ (Kamiński-Pstrucha 2019)
- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- chair or cricket (Jooken 2026+)
- $\overline{P_5}$ (Dhaliwal-Hamel-Hoàng-Maffray-McConnel-Panait 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)
- dart (Xia-Jooken-Goedgebeur-Huang 2023)
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$ (Xia-Jooken-Goedgebeur-Huang 2024)
- $W_4$ (Xia-Jooken-Goedgebeur-Beaton-C.-Huang 2025)
- bull (Belavadi-Hoàng 2026+)

**Theorem (C.-Hoàng 2023):** If $G$ is a $k$-critical $(P_5,$ gem$)$-free graph, then $G$ is a clique blowup of one of the graphs below (and therefore finitely many): (gem=  ) Proof is mostly pen-and-paper-style mathematics



(a) $G_1$ (b) $G_2$ (c) $G_3$ (d) $G_4$ (e) $G_5$



(f) $G_6$ (g) $G_7$ (h) $G_8$ (i) $G_9$ (j) $G_{10}$

Theoretical Computer Science 961 (2023) 113936

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# A refinement on the structure of vertex-critical ($P_5$, gem)-free graphs ☆

Ben Cameron [a,*], Chính T. Hoàng [b]

[a] Department of Computing Science, The King's University, Edmonton, AB, Canada
[b] Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, ON, Canada

The number of $k$-critical ($P_5$, gem)-free graph is exactly:

- 3 when $k = 4$
- 7 when $k = 5$
- 19 when $k = 6$
- 46 when $k = 7$

Graphs and Combinatorics (2024) 40:30
https://doi.org/10.1007/s00373-024-02756-x

ORIGINAL PAPER

# Infinite Families of $k$-Vertex-Critical $(P_5, C_5)$-Free Graphs

Ben Cameron[1] · Chính Hoàng[2]

Received: 4 July 2023 / Revised: 4 January 2024 / Accepted: 17 January 2024


**Theorem (C.-Hoàng 2024):** $G(q, k-1)$ is $k$-critical $(P_5, C_5)$-free graphs for all $q \geq 1$ and $k \geq 6$. Thus, there are infinitely many such graphs. Proof is pure pen-and-paper math, but lots of code used for generating the ideas!

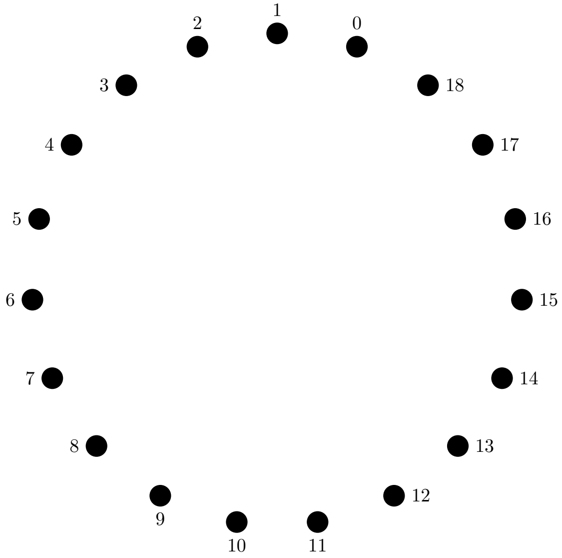
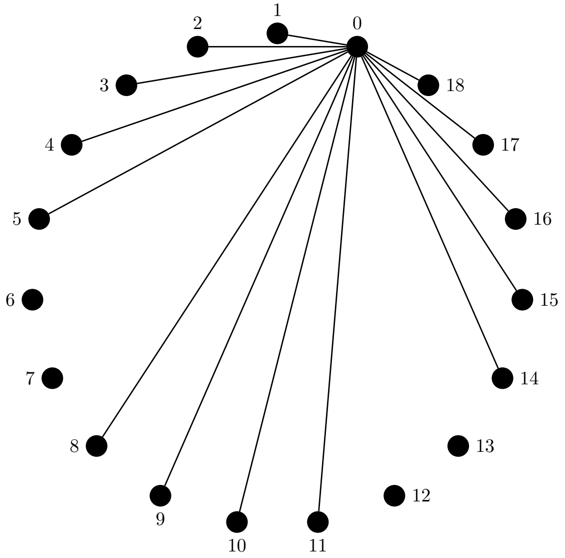



Figure: Constructing and colouring the 7-critical graph $G(3, 6)$.

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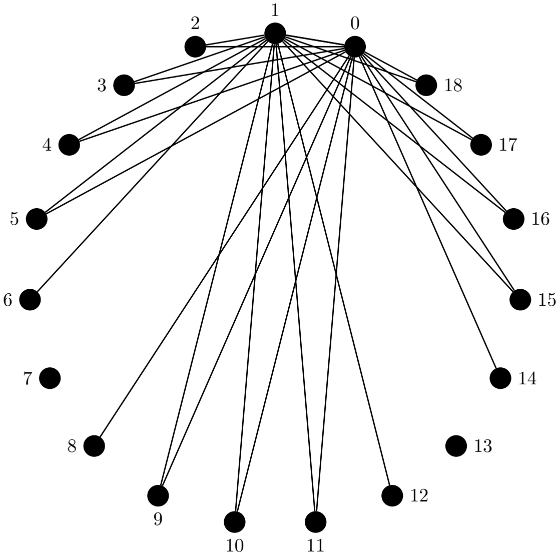


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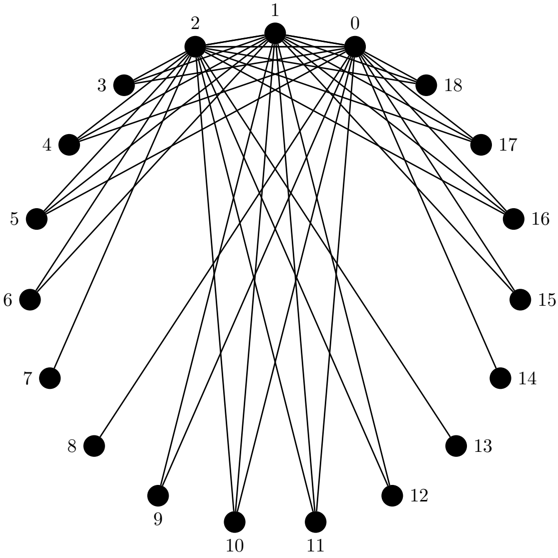


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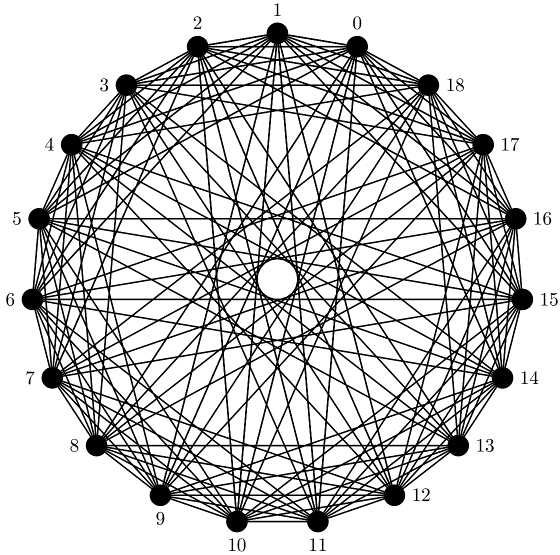


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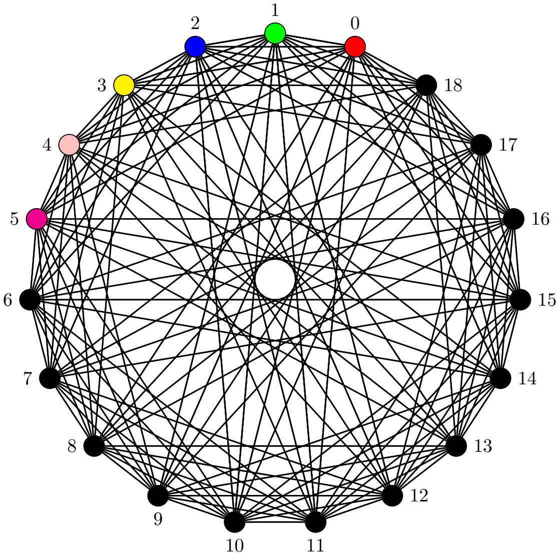


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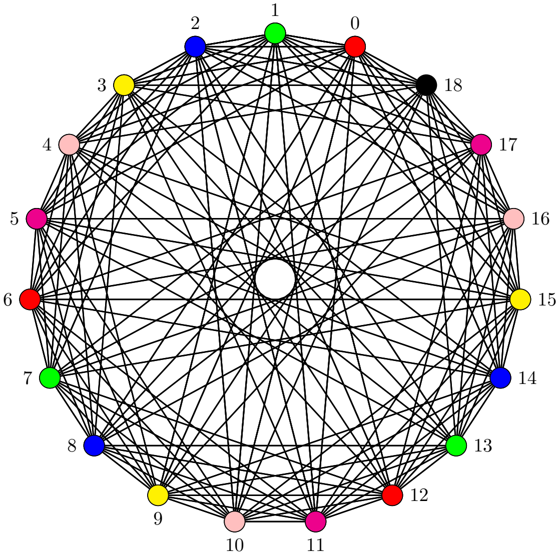


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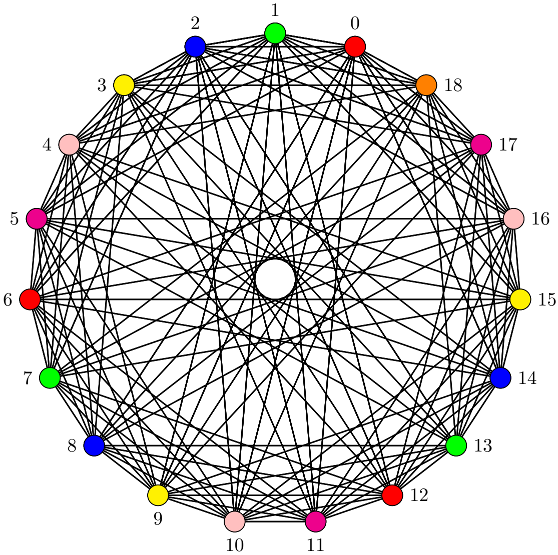


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Theoretical Computer Science 1075 (2026) 115934

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Vertex-critical $(P_5, W_4)$-free graphs ★

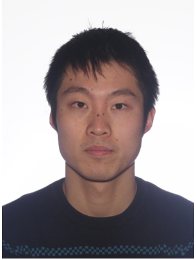
Wen Xia[a], Jorik Jooken[b], Jan Goedgebeur[b,c], Iain Beaton[d], Ben Cameron[e], Shenwei Huang[f,*]

[a] School of Mathematics and Statistics, Ankang University, Ankang, 725000, China
[b] Department of Computer Science, KU Leuven Campus Kulak-Kortrijk, Kortrijk, 8500, Belgium
[c] Department of Mathematics, Computer Science and Statistics, Ghent University, Ghent, 9000, Belgium
[d] Department of Mathematics and Statistics, Acadia University, Wolfville, NS, Canada
[e] School of Mathematical and Computational Sciences, University of Prince Edward Island, Charlottetown, PE, Canada
[f] School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, China

The number of $k$-critical $(P_5, W_4)$-free graph is exactly:

- 11 when $k = 4$
- 64 when $k = 5$

(a) Wen Xia



(b) Shenwei Huang

(c) Jorik Jooken

(d) Jan Goedgebeur

(e) Iain Beaton

**Theorem (Xia-Jooken-Goedgebeur-Beaton-C.-Huang 2025):** For every fixed integer $k \geq 1$, there are only finitely many $k$-critical $(P_5, W_4)$-free graphs.

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- If $G$ is $k$-critical and perfect, then $G = K_k$.

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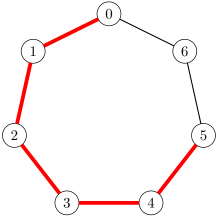
- If $G$ is $k$-critical and perfect, then $G = K_k$.
- Every $P_5$-free graph is also $C_{2k+1}$-free for all $k \geq 3$.



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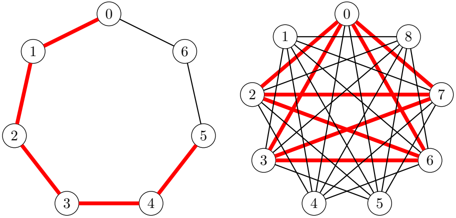

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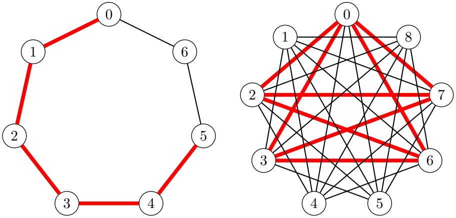
- Every $P_5$-free graph is also $C_{2k+1}$-free for all $k \geq 3$.
- Every $W_4$-free graph is also $\overline{C_{2k+1}}$-free for all $k \geq 4$.
- By SPGT, a $k$-critical $(P_5, W_4)$-free graph must be $K_k$ or contain an induced $C_5$ or $\overline{C_7}$.



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Proof Ideas (cont.): Build from an induced $C_5$ and $\overline{C_7}$, partitioning the rest of the vertices by their neighbourhoods. Then determine which of these sets are:

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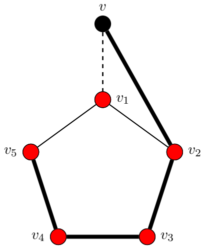
- Empty



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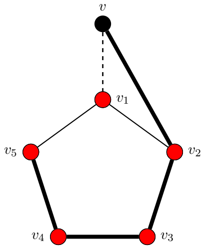
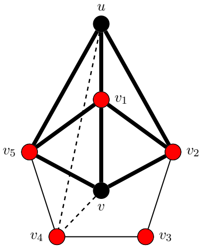
- Empty
- Cliques



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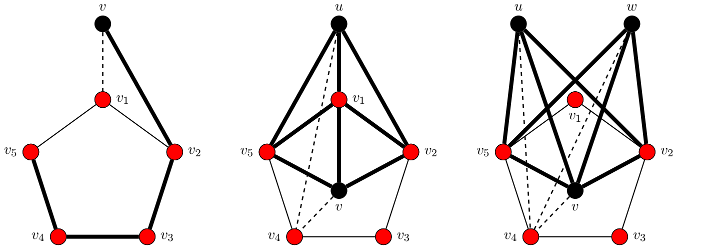
Proof Ideas (cont.): Build from an induced $C_5$ and $\overline{C_7}$, partitioning the rest of the vertices by their neighbourhoods. Then determine which of these sets are:

- Empty
- Cliques
- $P_3$-free



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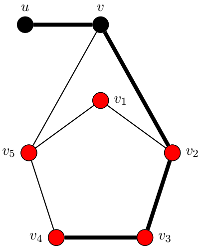

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Proof Ideas (cont.): Of these nonempty sets, determine which pairs are

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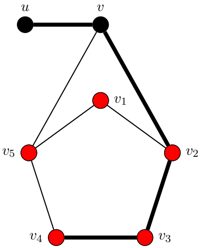
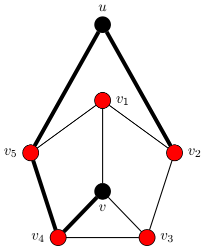
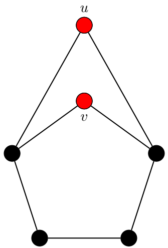
- Complete



Proof Ideas (cont.):

**Lemma** (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): Let $G$ be a $k$-critical graph. Then $G$ has no two nonadjacent vertices $u$ and $v$ such that $N(u) \subseteq N(v)$.



Proof Ideas (cont.):

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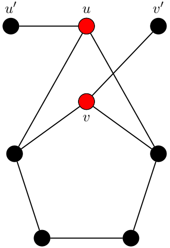

Proof Ideas (cont.):

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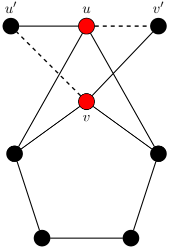


**Theorem (Beaton-C. 2026++):** There are only finitely many $k$-critical $(P_5, P_4 + \ell P_1)$-free graphs for all $k, \ell$.

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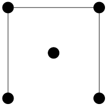
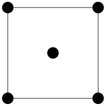
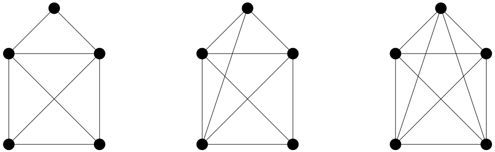

(f) $P_4 + P_1$


(g) $C_4 + P_1$

What's next? Handle the remaining cases to a dichotomy theorem on the finiteness of $k$-critical $(P_5, H)$-free graphs when $H$ is order 5:



(h) $\overline{P_3 + 2P_1}$ (i) $K_5 - e$ (j) $K_5$

Figure: Graphs $H$ of order 5 where the finiteness of $k$-critical $(P_5, H)$-free graphs is unknown.

# Thank You!