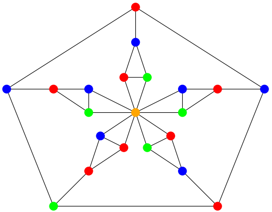
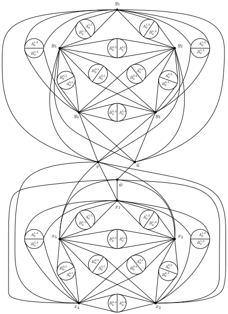
On Gyárfás' Path-Colour Problem

Ben Cameron

Belgian Graph Theory Conference 2025

Joint work with Alexander Clow (SFU)





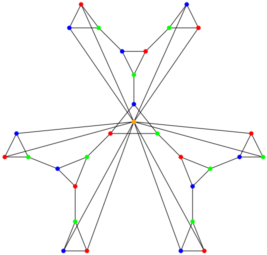


Figure: Alexander Clow, my co-author (a current PhD student at Simon Fraser University in Vancouver, BC, Canada) with his dog Rory.

Table of Contents

On Gyárfás' Path-Colour Problem

Problem (Hajnal-Erdős from Gyárfás in "Fruit Salad")

Is there a function f such that if each odd cycle of a graph spans a subgraph with chromatic number at most r, then the chromatic number of the graph is at most $f(r)$?

On Gyárfás' Path-Colour Problem

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4. Chromatic bound on cycles and paths.

Szentendre, 1995. One year had gone by but in Paul's book, like in Santa's sack,

there is always a new surprise. 'A problem with Hajnal: if each odd cycle of a graph G spans a subgraph with chromatic number at most r then the chromatic number of the graph is bounded by a function of r.'

Gyárfás' Conjecture

THE ELECTRONIC JOURNAL OF COMBINATORICS 4 (1997), #R8 8

Conjecture 2. *If each path of a graph spans at most 3-chromatic subgraph then the graph is c-colorable (with a constant c, perhaps with $c = 4$).*

We will let $r(G) = r$ denote the least integer r such that each path of G spans an r-colourable subgraph.

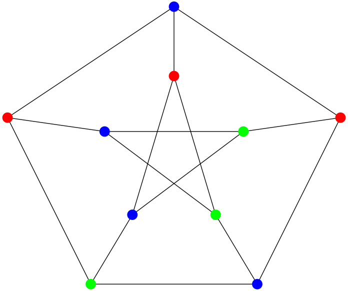


Table of Contents

Why must $c > 3$?

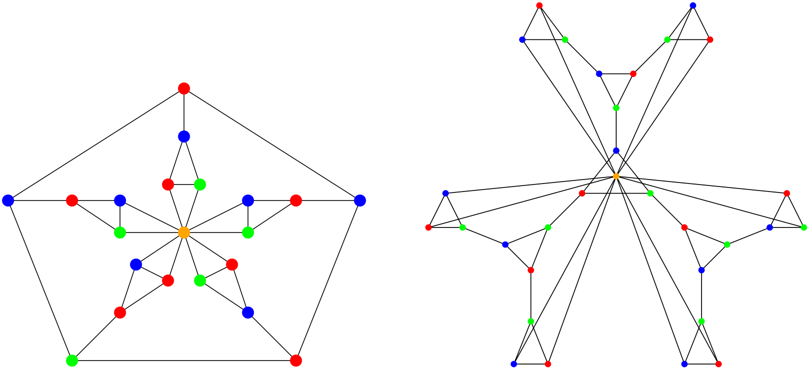
Theorem (Gallai 1963)

For all $c \geq 4$ there are infinitely many integers n for which there exists a k-critical graph G on n vertices such that no cycle of G has length at least $\frac{2(c-1)\log(n)}{\log(c-2)}$ and no path has length greater than $\frac{4(c-1)\log(n)}{\log(c-2)}$.

Corollary

For all integer $r \geq 3$, there exists a graph G with $r(G) = r$ and $\chi(G) \geq r+1$.

Examples with $r = 3$ and $\chi = 4$



Two graphs with $r = 3$ and chromatic number 4.
(Randerath-Schiermeyer 2002, left), (Gallai 1963, right)

Current Best

Theorem (Randerath and Schiermeyer 2002)

If G is an n-vertex graph with $r(G) \leq r$, then $\chi(G) \leq r \log_{\frac{8}{7}}(n)$.

Question

Is there a graph G with $r(G) \leq 3$ and $\chi(G) > 4$? If $r > 3$, does there exist a graph H with $r(H) \leq r$ and $\chi(H) > r + 1$?

Table of Contents

Theorem (Cameron,C. 2025+)

For all $r \geq 3$ there exists a graph G with $r(G) = r$ and

$$\chi(G) \geq \left\lfloor \frac{3r}{2} \right\rfloor - 1.$$

Corollary (Cameron,C. 2025+)

For all constants k there exists a graph G with

$$\chi(G) - r(G) > k.$$

The Construction for $r \geq 6$:

For each $r \geq 6$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor - 1$ we define a graph $G_k^{(r)}$ such that

$$\chi(G_k^{(r)}) = r + k \quad \text{and} \quad r(G_k^{(r)}) = r.$$

The Construction for $r \geq 6$:

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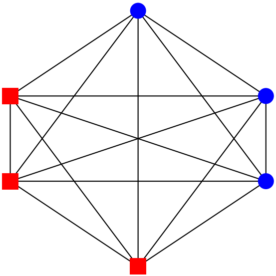
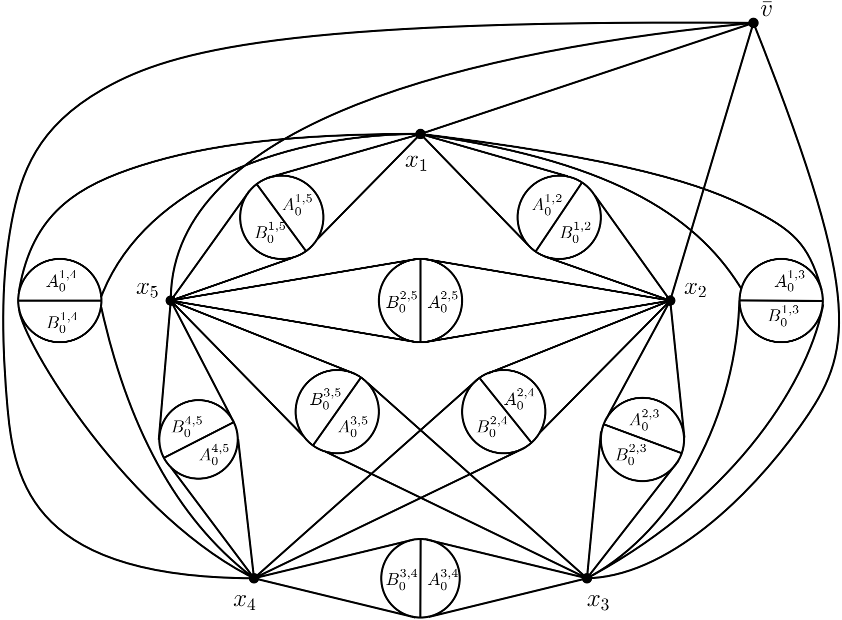
Let $G_0^{(r)} = K_r$.



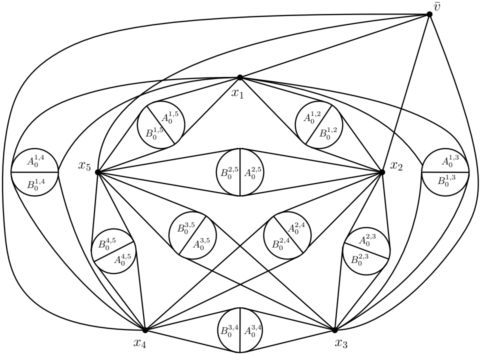
For each r and k we partition $V(G_k^{(r)})$ into two parts we call the A-set (blue circles) and B-set (red squares).



$\bar{v}$
x_1
x_2
x_3
x_4
x_5
$A_0^{1,5}$ $B_0^{1,5}$
$A_0^{1,2}$ $B_0^{1,2}$
$A_0^{1,4}$ $B_0^{1,4}$
$B_0^{2,5}$ $A_0^{2,5}$
$A_0^{1,3}$ $B_0^{1,3}$
$B_0^{3,5}$ $A_0^{3,5}$
$A_0^{2,4}$ $B_0^{2,4}$
$B_0^{4,5}$ $A_0^{4,5}$
$A_0^{2,3}$ $B_0^{2,3}$
$B_0^{3,4}$ $A_0^{3,4}$

From $G_k^{(r)}$ to $G_{k+1}^{(r)}$:

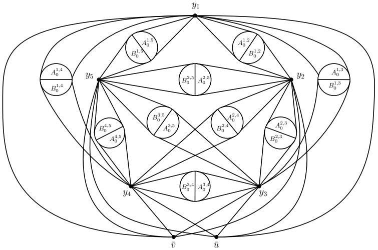
Step.1: Take an independent set $X = \{x_1, \ldots, x_{r+k-1}\}$ add a vertex $\bar{v}$ whose neighbourhood is X, and for all x_i, x_j add a copy of $G_k^{(r)}$ whose only neighbours in $X \cup \bar{v}$ are x_i and x_j. x_i is adjacent to the A-set, x_j is adjacent to the B-set.

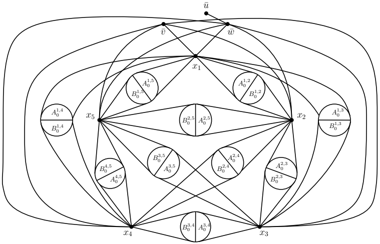


Taking $r = 6$ and $k = 0$.

From $G_k^{(r)}$ to $G_{k+1}^{(r)}$:

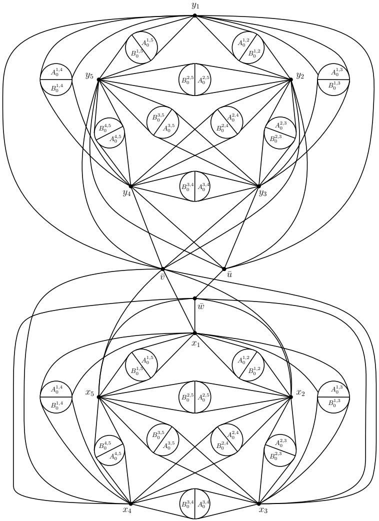
Step.2:





From $G_k^{(r)}$ to $G_{k+1}^{(r)}$:

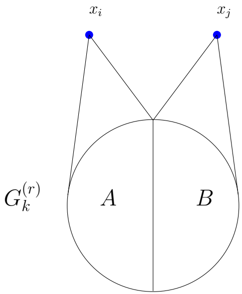
Step.3:



Why does $\chi(G_k^{(r)})$ increase with k?

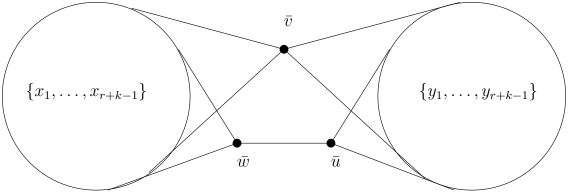
Step.1: Suppose $\chi(G_k^{(r)}) = r + k$.

Step.2: Vertices x_i, x_j in $G_{k+1}^{(r)}$ must receive different colours in an $(r+k)$-colouring of $G_k^{(r)}$.



Why does $\chi(G_k^{(r)})$ increase with k?

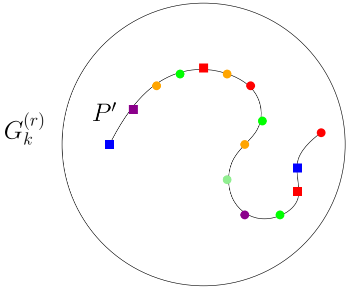
Step.3: Thus, in an $(r+k)$-colouring $\bar{v}, \bar{w}, \bar{u}$ receive the same colour.



$\therefore \chi(G_{k+1}^{(r)}) \geq r + k + 1.$

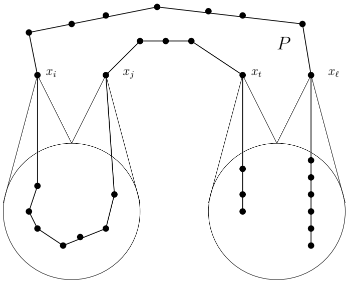
Why is $r(G_k^{(r)}) = r$?

Step.1: Suppose for every path P' in $G_k^{(r)}$ there is an r-colouring of $G_k^{(r)}[P']$ where the A-set gets private colours and the B-set gets private colours.



Why is $r(G_k^{(r)}) = r$?

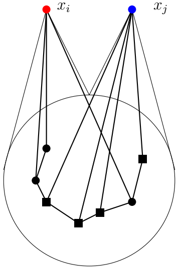
Step.2: Consider how a path P in $G_{k+1}^{(r)}$ intersects a copy of $G_k^{(r)}$ adjacent to x_i, x_j.



Step.3: Now we assign x_i, x_j different colours if P intersects the copy of $G_k^{(r)}$ adjacent to x_i, x_j. This requires at most 4 colours.

Why is $r(G_k^{(r)}) = r$?

Step.4: Once vertices x_i, x_j are coloured we can extend to the copies of $G_k^{(r)}$ adjacent to x_i, x_j.



We lose control of 1 private A-colour and 1 private B-colour.

Table of Contents

THE ELECTRONIC JOURNAL OF COMBINATORICS 4 (1997), #R8 8

Conjecture 2. *If each path of a graph spans at most 3-chromatic subgraph then the graph is c-colorable (with a constant c, perhaps with $c = 4$).*

Conjecture (C.-Clow 2025+)

For every constant $c > 0$, there exists a graph G with $\chi(G) \geq c \cdot r(G)$.

Theorem (C.-Clow 2025+)

If H is a forest on at most 4 vertices not isomorphic to the claw, then every H-free graph G has $\chi(G) \leq r(G) + 1$.

Conjecture (C.-Clow 2025+)

If G is a P_5-free graph, then $\chi(G) = r(G)$.

Theorem (C.-Clow 2025+)

If H is a forest on at most 4 vertices not isomorphic to the claw, then every H-free graph G has $\chi(G) \leq r(G) + 1$.

Conjecture (C.-Clow 2025+)

If G is a P_5-free graph, then $\chi(G) = r(G)$.

Lemma (C.-Clow 2025+)

If G is a graph with $\alpha(G) \leq 3$, then $\chi(G) = r(G)$.

Problem (C.-Clow 2025+)

For all graphs G such that $\alpha(G) \leq 4$, is $\chi(G) = r(G)$?

Theorem (C.-Clow 2025+)

Let $t \geq 4$ be an integer. If G is a $K_{1,t}$-free graph and $r(G) \leq r$, then

$$\chi(G) \leq (t-1)\left(r + \binom{t-1}{2} - 3\right).$$

If G is claw-free, then $\chi(G) \leq 2r(G)$.

Question (C.-Clow 2025+)

What is the least integer t such that there exists a $K_{1,t}$-free graph G with $\chi(G) > r(G)$?

Thank You!

Figure: Scan QR code for our preprint. Thanks to NSERC!