Vertex-critical $(P_3 + \ell P_1)$-free graphs

Ben Cameron (he/him)

The King's University

ben.cameron@kingsu.ca

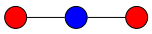
(Joint work with Tala Abuadas, Chính Hoàng, and Joe Sawada)

Coast Combinatorics Conference 2023

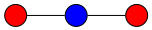
March 4, 2023

Figure: Ben, Asher, Joe, Kaito - Toronto, ON 2021

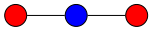
Definitions

- P_n is the path on n vertices (P_3: ).

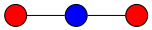
Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.

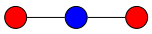
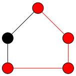
Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).

Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Coloring here means proper coloring (adjacent vertices get different colors).

Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Coloring here means proper coloring (adjacent vertices get different colors).
- A graph is H-free if it does not contain H as an induced subgraph.  is P_5-free but not P_4-free.

Definition: For fixed k, the k-Coloring decision problem is to determine if a given graph is k-colorable.

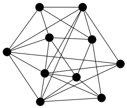


Figure: Decide 3-Coloring for this graph.

Definition: For fixed k, the k-Coloring decision problem is to determine if a given graph is k-colorable.

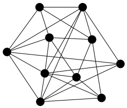


Figure: Decide 3-Coloring for this graph.

- k-Coloring is NP-complete for all $k \geq 3$ (Karp 1972).

Definition: For fixed k, the k-COLORING decision problem is to determine if a given graph is k-colorable.

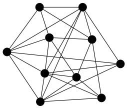


Figure: Decide 3-COLORING for this graph.

- k-COLORING is NP-complete for all $k \geq 3$ (Karp 1972).
- It remains NP-complete when restricted to H-free graphs if H contains a cycle (Kamiński-Lozin 2007).

Definition: For fixed k, the k-Coloring decision problem is to determine if a given graph is k-colorable.

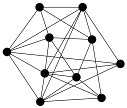
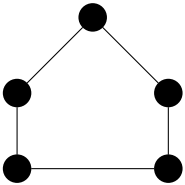


Figure: Decide 3-Coloring for this graph.

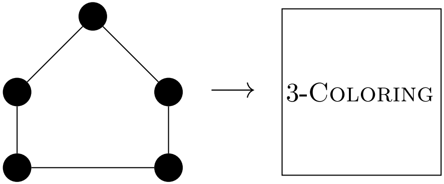
- k-Coloring is NP-complete for all $k \geq 3$ (Karp 1972).
- It remains NP-complete when restricted to H-free graphs if H contains a cycle (Kamiński-Lozin 2007).
- It remains NP-complete when restricted to H-free graphs if H contains a claw (Hoyler 1981; Leven-Gail 1983).

Theorem (Hoàng et al. 2010) k-Coloring P_5-free graphs can be solved in polynomial-time for all k

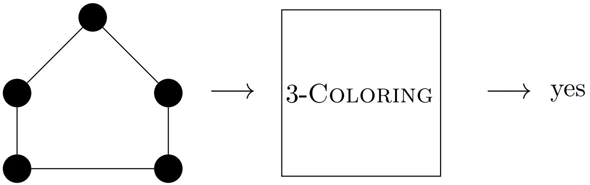
Theorem (Hoàng et al. 2010) k-Coloring P_5-free graphs can be solved in polynomial-time for all k



Theorem (Hoàng et al. 2010) k-Coloring P_5-free graphs can be solved in polynomial-time for all k



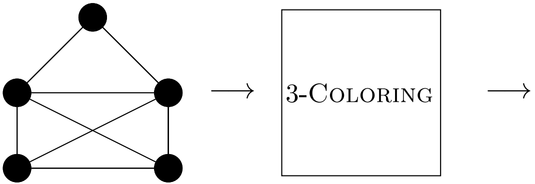
Theorem (Hoàng et al. 2010) k-Coloring P_5-free graphs can be solved in polynomial-time for all k



Theorem (Hoàng et al. 2010) k-COLORING P_5-free graphs can be solved in polynomial-time for all k and the algorithm gives a valid k-coloring if one exists.

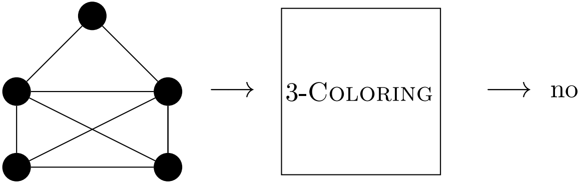
- A k-coloring is a certificate to verify a “yes”.

Theorem (Hoàng et al. 2010) k-COLORING P_5-free graphs can be solved in polynomial-time for all k and the algorithm gives a valid k-coloring if one exists.



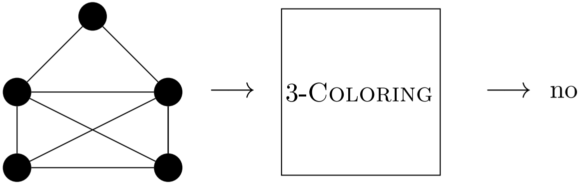
- A k-coloring is a certificate to verify a "yes".

Theorem (Hoàng et al. 2010) k-COLORING P_5-free graphs can be solved in polynomial-time for all k and the algorithm gives a valid k-coloring if one exists.



- A k-coloring is a certificate to verify a "yes".

Theorem (Hoàng et al. 2010) k-COLORING P_5-free graphs can be solved in polynomial-time for all k and the algorithm gives a valid k-coloring if one exists.



- A k-coloring is a certificate to verify a "yes".
- How can we verify a "no"?

- A graph G is k-vertex-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.

- A graph G is k-vertex-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.
- Every graph that is not k-colorable has a $(k+1)$-vertex-critical induced subgraph.

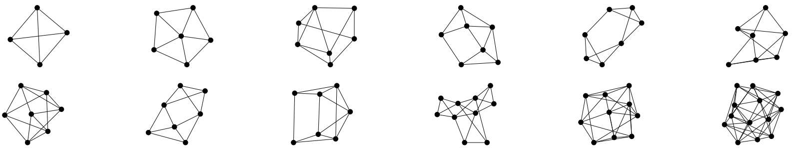
- A graph G is k-vertex-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.
- Every graph that is not k-colorable has a $(k+1)$-vertex-critical induced subgraph.

Issue 1: For $k \geq 3$ there are an infinite number of k-vertex-critical graphs.

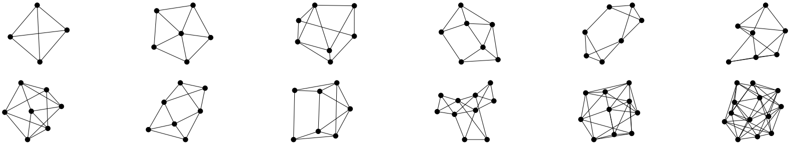
- A graph G is k-vertex-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.
- Every graph that is not k-colorable has a $(k+1)$-vertex-critical induced subgraph.

Issue 1: For $k \geq 3$ there are an infinite number of k-vertex-critical graphs.
Issue 2: k-vertex-critical is a mouthful, so simply k-critical from now on.

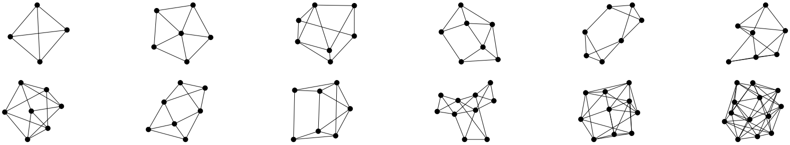
Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.

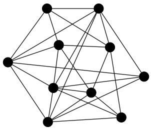


Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.

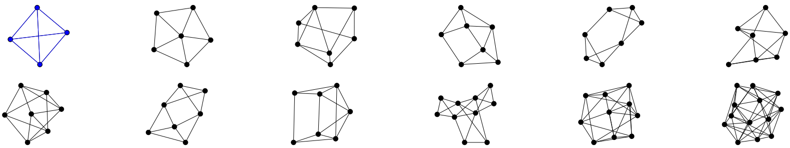


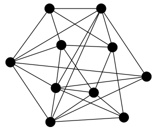
Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.



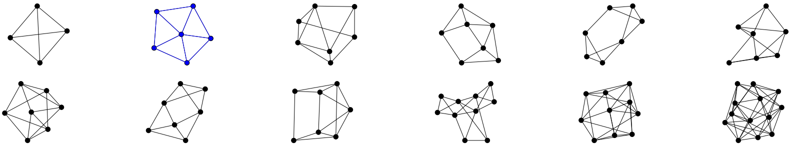


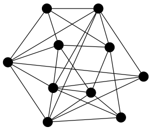
Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.



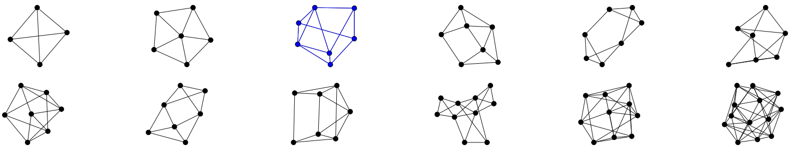


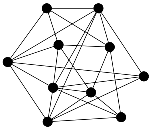
Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.



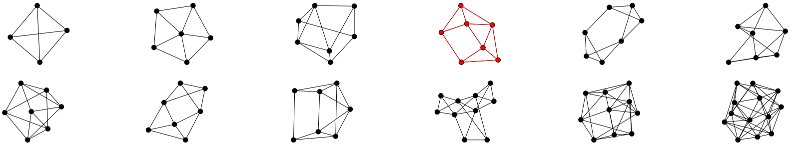


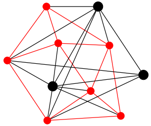
Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.





Theorem (Bruce et al. 2009, Maffray-Morel 2012) All twelve 4-critical P_5-free graphs are given below.





More bad news

Theorem (e.g. Erdős 1959): If H contains an induced cycle, then there is an infinite number of k-critical H-free graphs for all $k \geq 3$.

Theorem (e.g. Lazebnik-Ustimenko 1995): If H contains an induced claw, then there is an infinite number of k-critical H-free graphs for all $k \geq 3$.

More bad news

Theorem (e.g. Erdős 1959): If H contains an induced cycle, then there is an infinite number of k-critical H-free graphs for all $k \geq 3$.

Theorem (e.g. Lazebnik-Ustimenko 1995): If H contains an induced claw, then there is an infinite number of k-critical H-free graphs for all $k \geq 3$.

Theorem (Hoàng et al. 2015): If H contains an induced $2P_2$, then there is an infinite number of k-critical H-free graphs for all $k \geq 5$.

Question For which H are there only finitely many k-critical H-free graphs for all k?

Question For which H are there only finitely many k-critical H-free graphs for all k?

Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are only finitely many 4-critical H-free graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + \ell P_1$ for some $\ell \geq 0$.

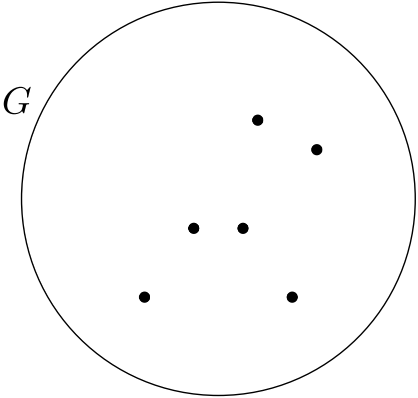
Question For which H are there only finitely many k-critical H-free graphs for all k?

Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are only finitely many 4-critical H-free graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + \ell P_1$ for some $\ell \geq 0$.

H must be one of:

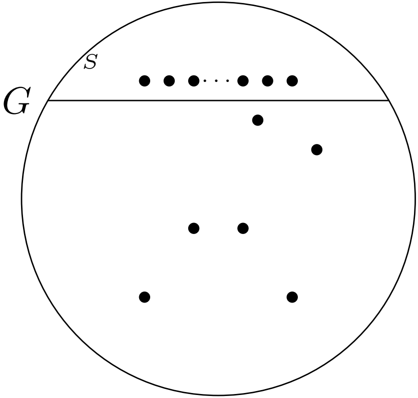
- ℓP_1
- $P_2 + \ell P_1$
- $P_3 + \ell P_1$
- $P_4 + \ell P_1$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



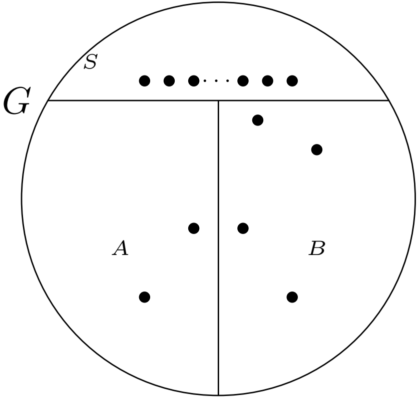
- G:k-critical $(P_3 + \ell P_1)$-free

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



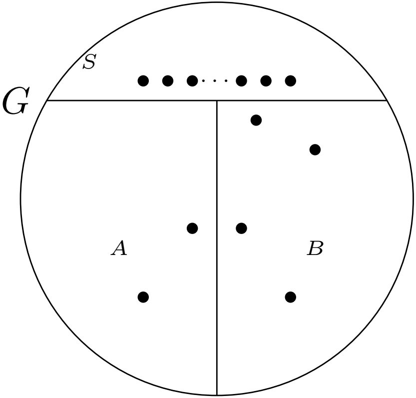
- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



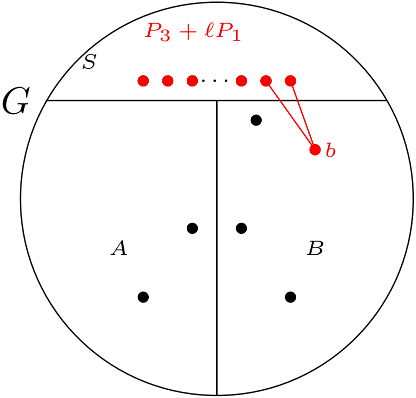
- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



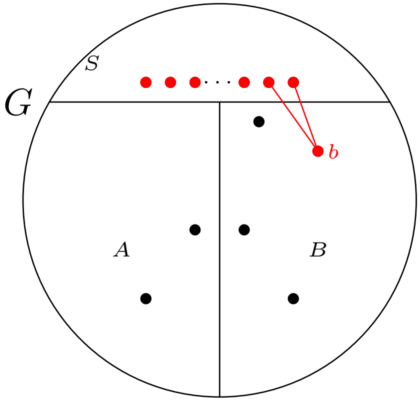
- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$
- $B = \{v \in G - S : |N(v) \cap S| \geq 2\}$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$
- $B = \{v \in G - S : |N(v) \cap S| \geq 2\}$
- $\forall b \in B \ \ |N(b) \cap S| \geq |S| - \ell + 1$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.

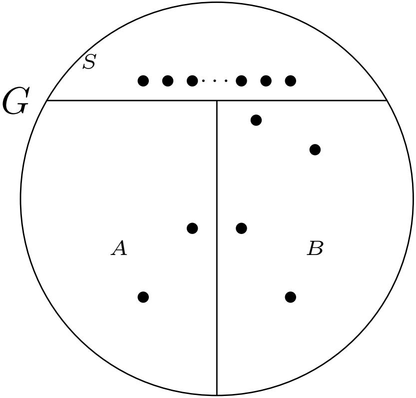


- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$
- $B = \{v \in G - S : |N(v) \cap S| \geq 2\}$
- $\forall b \in B \ \ |N(b) \cap S| \geq |S| - \ell + 1$

$\vdots$

- Color c appears in $B \Rightarrow$ at most $(k-1)(\ell+3) - 1$ vertices get c

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



- G:k-critical $(P_3 + \ell P_1)$-free
- S:maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$
- $B = \{v \in G - S : |N(v) \cap S| \geq 2\}$
- $\forall b \in B \ \ |N(b) \cap S| \geq |S| - \ell + 1$

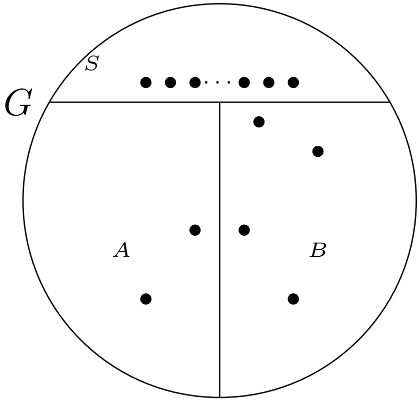
$\vdots$

- Color c appears in $B \Rightarrow$ at most $(k-1)(\ell+3) - 1$ vertices get c

$\vdots$

- $\alpha(G) < (k-1)^2(\ell+3)$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.



- G: k-critical $(P_3 + \ell P_1)$-free
- S: maximum independent set
- $A = \{v \in G - S : |N(v) \cap S| = 1\}$
- $B = \{v \in G - S : |N(v) \cap S| \geq 2\}$
- $\forall b \in B \;\; |N(b) \cap S| \geq |S| - \ell + 1$

$\vdots$

- Color c appears in $B \Rightarrow$ at most $(k-1)(\ell+3) - 1$ vertices get c

$\vdots$

- $\alpha(G) < (k-1)^2(\ell+3)$
- $|V(G)| < R(k, (k-1)^2(\ell+3))$ $\square$

Question Revisited

Question For which H are there only finitely many k-critical H-free graphs for all k?

H must be one of:

- ℓP_1
- $P_2 + \ell P_1$
- $P_3 + \ell P_1$
- $P_4 + \ell P_1$

Question Revisited

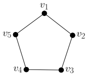
Question For which H are there only finitely many k-critical H-free graphs for all k?

H must be one of:

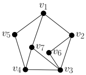
- ℓP_1
- $P_2 + \ell P_1$
- $P_3 + \ell P_1$
- $P_4 + \ell P_1$

Figure: The graphs gem (left) and co-gem=$P_4 + P_1$ (right).

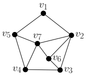
Theorem (Karthick-Maffray 2018): If G is (gem,co-gem)-free, then either G is perfect, or G is a P_4-free expansion of G_i for some $i \in \{1, 2, \ldots, 10\}$, or $G \in \mathcal{H}$.



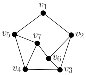
(a) G_1



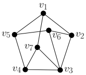
(b) G_2



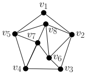
(c) G_3



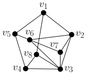
(d) G_4



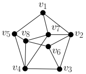
(e) G_5



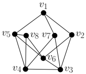
(f) G_6



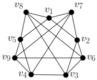
(g) G_7



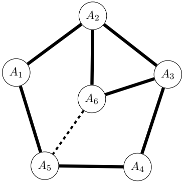
(h) G_8



(i) G_9



(j) G_{10}



(k) Graphs in $\mathcal{H}$

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical (gem, co-gem)-free graphs for all k and every such non-complete graph is a clique-expansion of C_5.

Theorem (Abuadas-C.-Hoàng-Sawada 2022+) There are only finitely many k-critical (gem, co-gem)-free graphs for all k and every such non-complete graph is a clique-expansion of C_5.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
num(k)	1	1	2	2	4	6	11	17	27	39	58	80	112	148	197	253

Table: num(k) denotes the number of k-critical (gem, co-gem)-free graphs.

Question For which values of $\ell \geq 1$ are there only finitely many k-critical $(P_4 + \ell P_1)$-free graphs for all k?

Question For which graph H are there only finitely many k-critical (P_5, H)-free graphs for all k?

Question For which values of $\ell \geq 1$ are there only finitely many k-critical $(P_4 + \ell P_1)$-free graphs for all k?

Question For which graph H are there only finitely many k-critical (P_5, H)-free graphs for all k?

Figure: Scan QR code for the paper. Thanks to NSERC for support!

Thank You!