A finite and an infinite family of k-critical (P_5, H)-free graphs

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(Joint work with Chính Hoàng)

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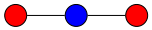
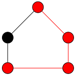
Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Coloring here means proper coloring (adjacent vertices get different colors).
- A graph is H-free if it does not contain H as an induced subgraph.  is P_5-free but not P_4-free.

Definition: For fixed k, the k-Coloring decision problem is to determine if a given graph is k-colorable.

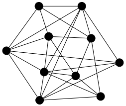


Figure: Decide 3-Coloring for this graph.

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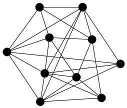

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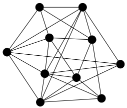

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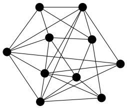

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- It remains NP-complete when restricted to H-free graphs if H contains a claw (Holyer 1981; Leven-Gail 1983).

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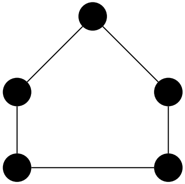

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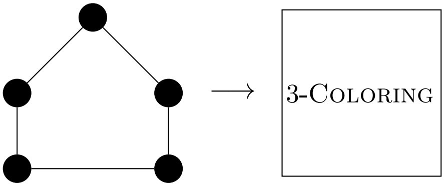

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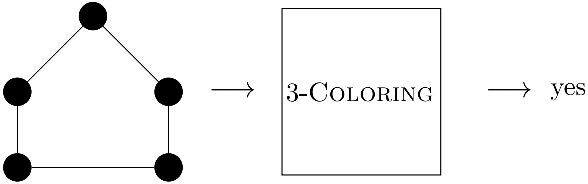

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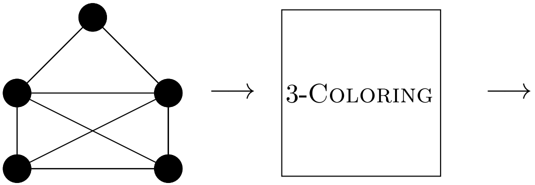

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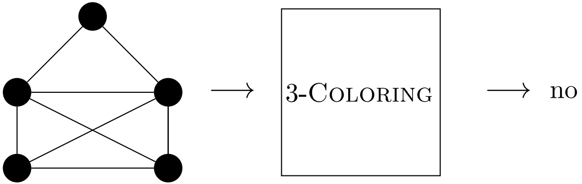

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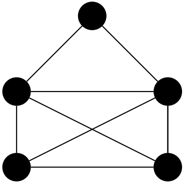

$\longrightarrow$ 3-COLORING $\longrightarrow$ no

- A k-coloring is a certificate to verify a "yes".
- How can we verify a "no"?

- A graph G is k-critical if G is not $(k-1)$-colorable, but every induced subgraph of G is.

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Issue: For $k \geq 3$ there are infinitely many k-critical graphs.

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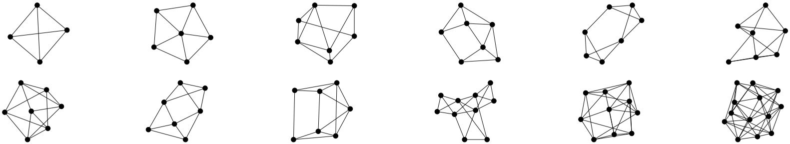

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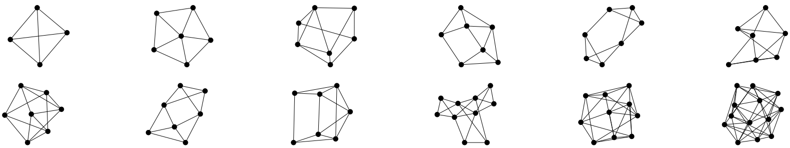

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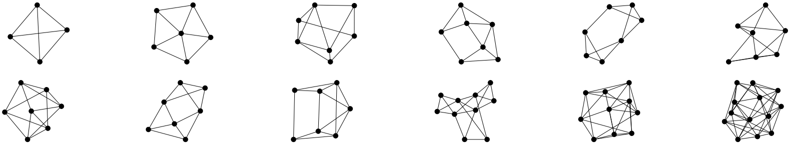

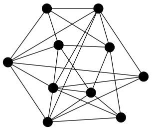

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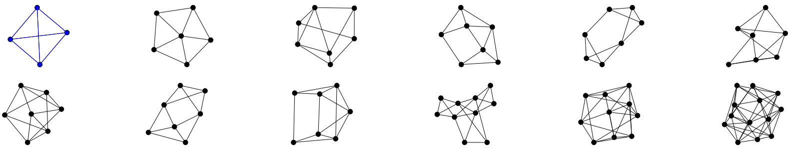

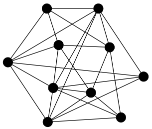

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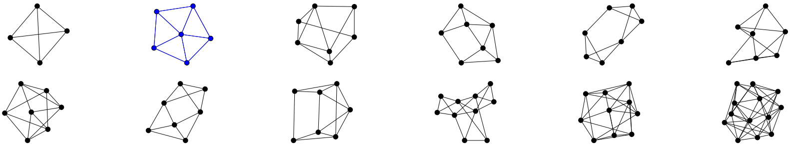

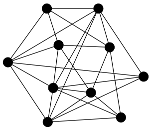

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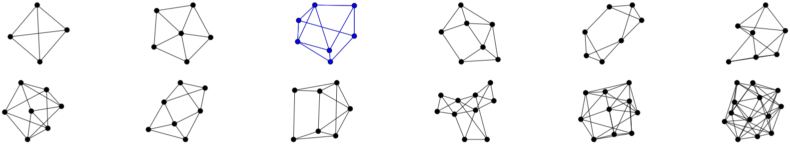

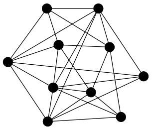

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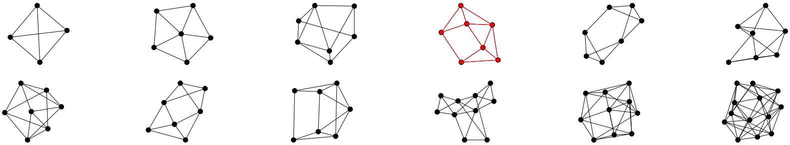

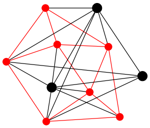

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Also finite if H is any from the list below:

- banner
- $K_{2,3}$ or $K_{1,4}$
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- $\overline{P_5}$
- $\overline{P_3 + P_2}$ or gem

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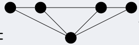
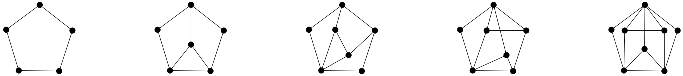
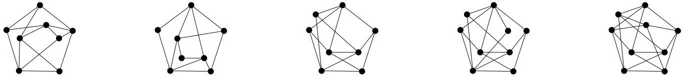
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2023+)
- $\overline{P_5}$ (Dhaliwal et al. 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)

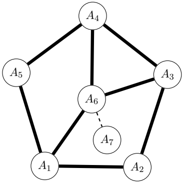
Theorem (Chudnovsky-Karthick-Maceli-Maffray 2020): If G is a (P_5, gem)-free graph, then G is either perfect, in $\mathcal{G}_i$ for some $i \in \{1, \ldots, 10\}$, or $G \in \mathcal{H}$. (gem= )



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G_5



(f) G_6 (g) G_7 (h) G_8 (i) G_9 (j) G_{10}



(k) Graphs in $\mathcal{H}$

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The number of k-critical $(P_5,\ \text{gem})$-free graph is exactly:

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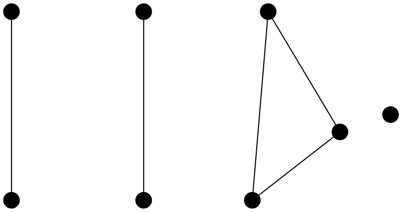

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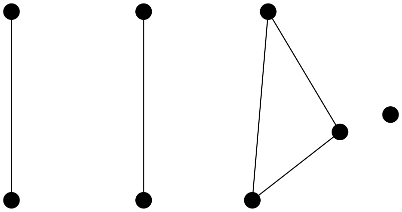
Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are infinitely many k-critical P_7-free for all $k \geq 4$.



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Theorem (Hoàng et al. 2015): There are only finitely many many 5-critical (P_5, C_5)-free graphs. (In fact, exactly 13)

Theorem (C.-Hoàng 2023+): $G(q, k-1)$ is k-critical (P_5, C_5)-free graphs for all $q \geq 1$ and $k \geq 6$. Thus, there are infinitely many such graphs.

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$$N(v_i) = \{v_{i-1}, v_{i+1}\} \cup \{v_{i+kj+m} : m = 2, 3, ...k-1 \text{ and } j = 0, 1, ..., q-1\}$$

where each index is taken modulo $kq + 1$.

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- When $q = 2$, $G(q, k) = \overline{C_{2k+1}}$
- When $k = 3$, $G(r, k) = G_r$ from Chudnovsky et al.'s 4-critical P_7-free family.

Theorem (C.-Hoàng 2023+): $G(q, k-1)$ is k-critical (P_5, C_5)-free graphs for all $q \geq 1$ and $k \geq 6$. Thus, there are infinitely many such graphs.

Let $G(q, k)$ be a graph on vertex set $\{v_0, v_1, ..., v_{kq}\}$ with

$$N(v_i) = \{v_{i-1}, v_{i+1}\} \cup \{v_{i+kj+m} : m = 2, 3, ...k-1 \text{ and } j = 0, 1, ..., q-1\}$$

where each index is taken modulo $kq + 1$.

- When $q = 1$, $G(q, k) = K_{k+1}$
- When $q = 2$, $G(q, k) = \overline{C_{2k+1}}$
- When $k = 3$, $G(r, k) = G_r$ from Chudnovsky et al.'s 4-critical P_7-free family.
- When $k = 4$, $G(p, k) = G_p$, from Hoàng et al.'s 5-critical P_5-free family.

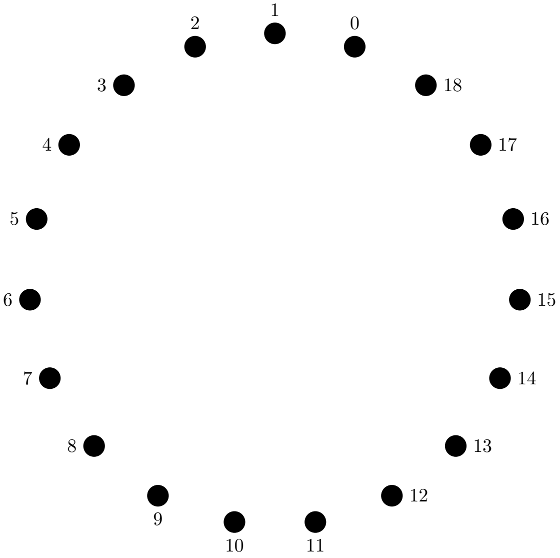
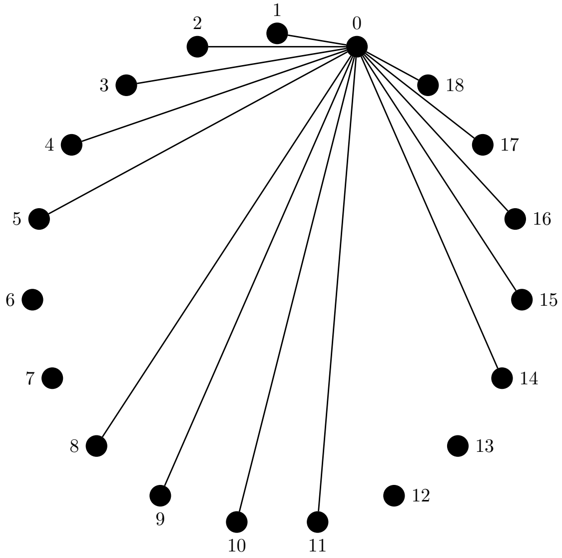


Figure: Constructing and colouring the 7-critical graph $G(3, 6)$.



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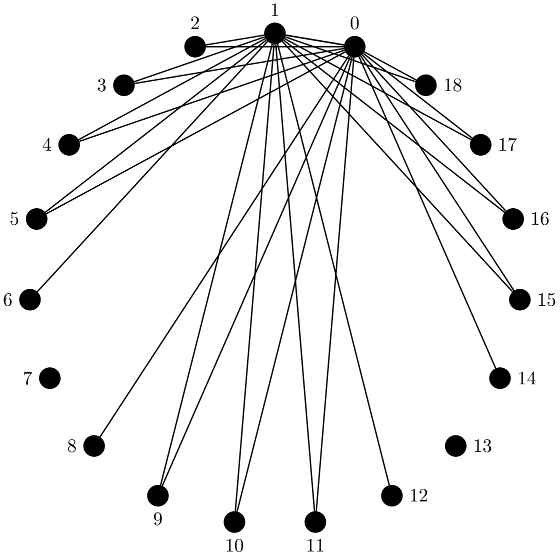

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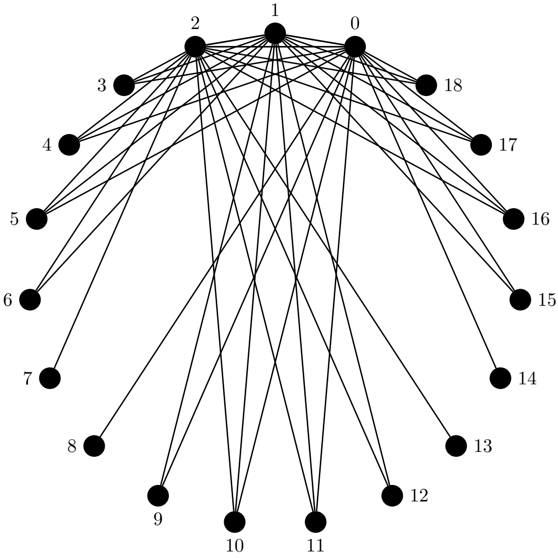

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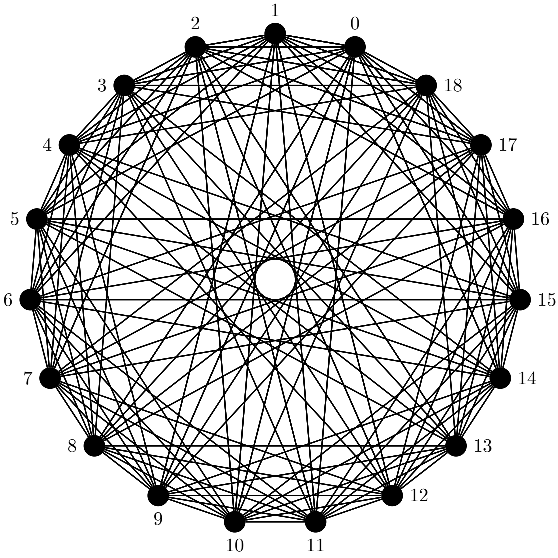

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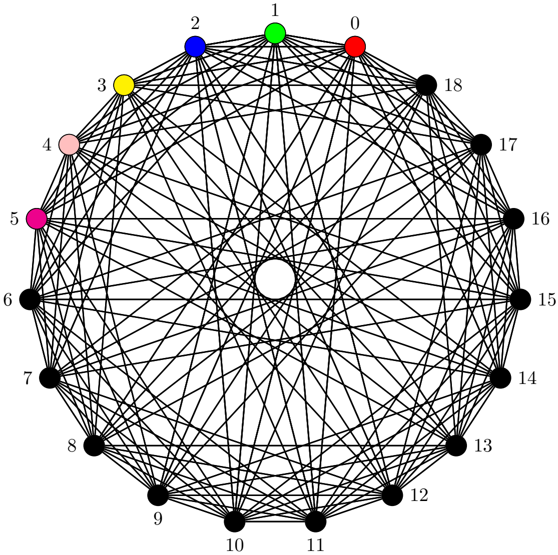

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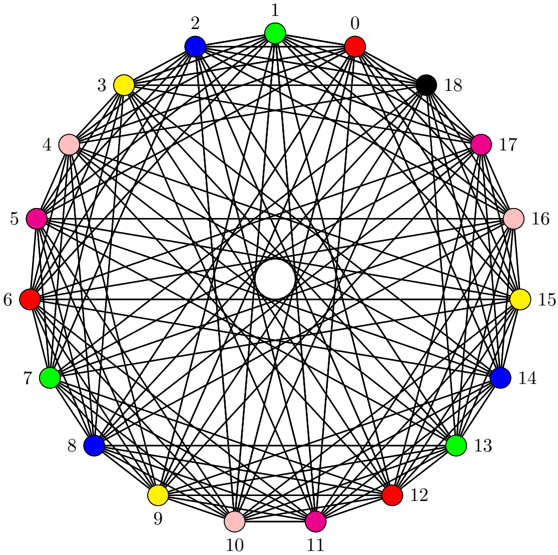

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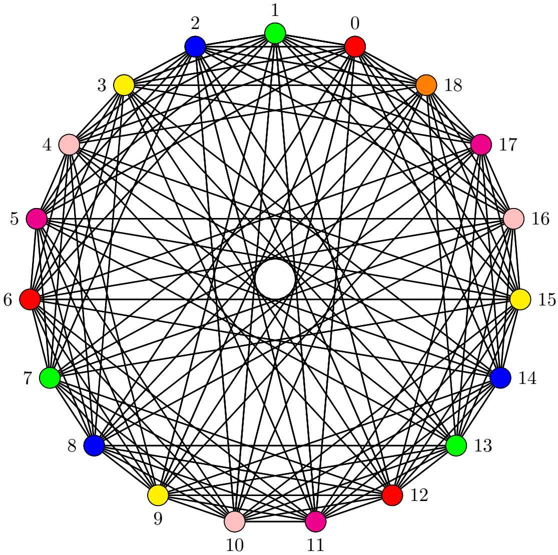

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In fact, $G(q, k-1)$ is actually $(2P_2, K_3 + P_1, C_5)$-free for all $k \geq 6$.

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Question For which graphs H are there only finitely many k-critical $(2P_2, K_3 + P_1, C_5, H)$-free graphs for all k?

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When H is order 5, only unknown for following graphs:

- claw+P_1
- $P_4 + P_1$
- chair (known $k = 5$)
- $\overline{\text{diamond} + P_1}$
- $C_4 + P_1$
- bull (known $k = 5$)
- dart
- $\overline{K_3 + 2P_1}$
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- W_4
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THANK YOU!

Figure: Scan QR code for (P_5, gem)-free paper. Thanks to NSERC!

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