On Gromov's Approximating Tree

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Approximating Trees

Definition

Let G be a graph. A *distance ϵ-approximating tree* of G is a tree with the same vertex set as G, such that for all $u, v \in G$,

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Many problems in graph theory are trivial on trees.

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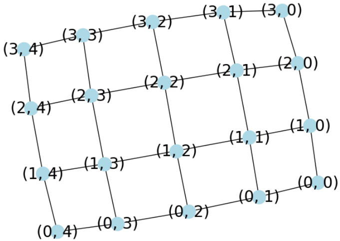
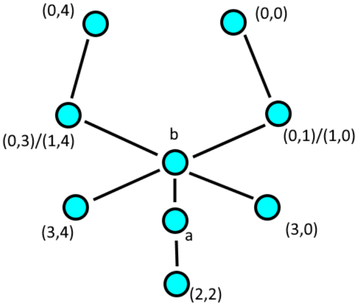
- T is non-distance-increasing, and
- T is a $2\delta \log(n)$-approximation of G, where δ is the *Gromov hyperbolicity* of G.

Combining these shows that, for all $u, v \in G$,

$$d_G(u, v) - 2\delta \log(n) \leq d_T(\varphi(u), \varphi(v)) \leq d_G(u, v).$$

Gromov's Tree





$$w = (2,2),\ \varphi^{-1}(a) = \{(1,2),(2,1),(2,3),(3,2)\},$$
$$\varphi^{-1}(b) = \{(0,2),(1,1),(1,3),(2,0),(2,4),(3,1),(3,3)\}$$

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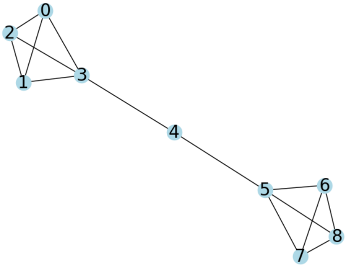

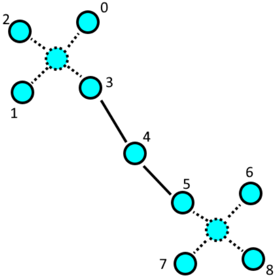

$$w = 0$$

Gromov's Tree

The proof of this theorem gives a general method of its construction, but with no mention of time complexity.

However, many articles cite these two works with claims that it can be done in $O(n^2)$ time. This turns out to be the case, if we start with the distance matrix D of our graph G.

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$$G \xrightarrow{O(n^{\omega} \log(n))} D \qquad \omega \approx 2.371552.$$

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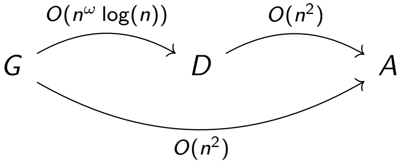

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What we have done is written an explicit algorithm that results in the following theorem.

Theorem (Cornect and Martinez-Pedroza, 2023)

There is an algorithm that takes as input the adjacency matrix of a graph G on n vertices, and outputs in time $O(n^2)$ the distance matrix A of an approximating tree, as described in Gromov's Theorem.

Gromov Hyperbolicity

Definition

In a metric space (X, d), the Gromov product of x and y with respect to z is given by

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Definition (four-point condition)

A space is called δ-hyperbolic if, for all w, x, y, $z \in X$,

$$\delta \geq \min\{(x|y)_w, (y|z)_w\} - (x|z)_w.$$

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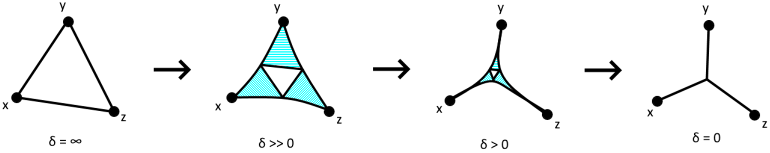

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Fournier et al. (2015) detailed a way of approximating δ using Gromov's approximating tree in $O(n^2)$ from D. Our algorithm allows us to do this directly from G, while staying $O(n^2)$.

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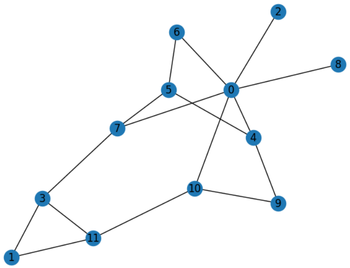
5. Repeat steps 3 and 4 for $d_w = \alpha - 2, \alpha - 3, \ldots, 0$.

An Example



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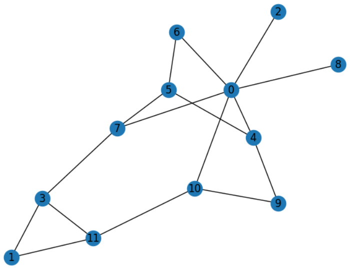

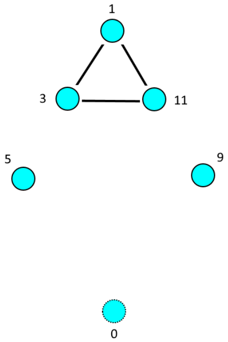

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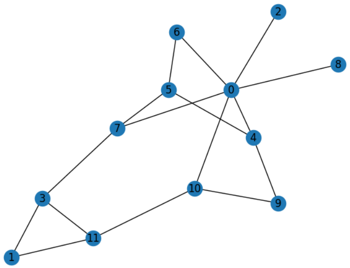

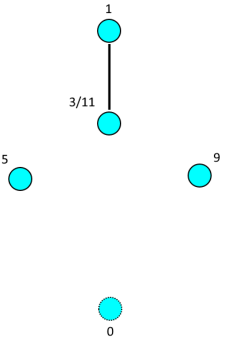

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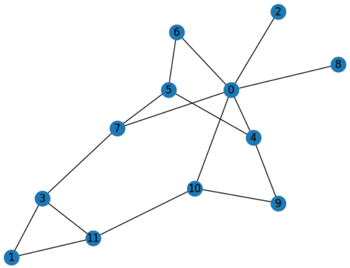

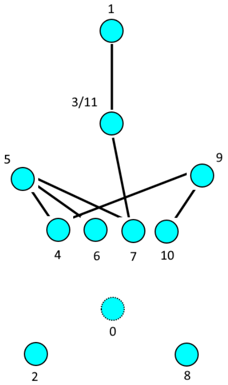

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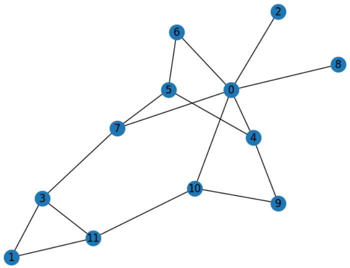

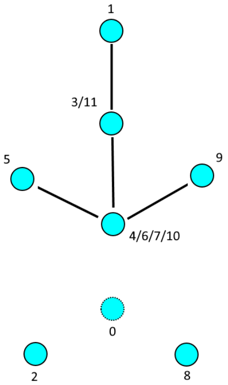

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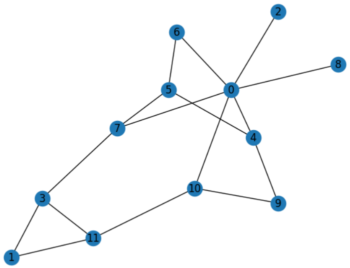

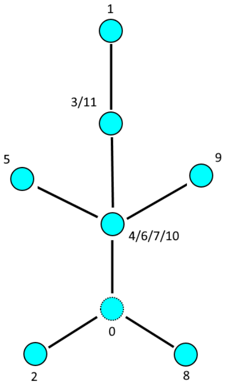

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- Is there a similar algorithm for strongly connected directed graphs (non-commutative metrics)?
- What is the connection between the All Pairs Bottleneck Problem (APBP) and Gromov's approximating tree?

Acknowledgements

Thank you!

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