The Proper Spherically Symmetric Frame for Teleparallel Geometries

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October 6, 2023

I. INTRODUCTION

It has been recently determined [1] that the appropriate frame ${h^a}_\mu$ and spin connection ω^a_{bc} ansatz containing the minimum number of arbitrary functions is of the form

$$
{h^a}_\mu = \begin{bmatrix} A_1(t,r) & 0 & 0 & 0 \\ 0 & A_2(t,r) & 0 & 0 \\ 0 & 0 & A_3(t,r) & 0 \\ 0 & 0 & 0 & A_3(t,r)\sin(\theta) \end{bmatrix} \tag{1}
$$

and

$$ {\omega^3}_{4t} = -\partial_t \chi, \qquad {\omega^3}_{4r} = -\partial_r \chi, \tag{2} $$

$$ {\omega^2}_{3\theta} = \cosh(\psi)\cos(\chi), \qquad {\omega^2}_{4\phi} = \cosh(\psi)\cos(\chi)\sin(\theta) \tag{3} $$

$$ {\omega^2}_{3\phi} = \cosh(\psi)\sin(\chi)\sin(\theta), \qquad {\omega^2}_{4\theta} = -\cosh(\psi)\sin(\chi) \tag{4} $$

$$ {\omega^1}_{2t} = \partial_t \psi, \qquad {\omega^1}_{2r} = \partial_r \psi, \tag{5} $$

$$ {\omega^1}_{3\theta} = -\sinh(\psi)\cos(\chi), \qquad {\omega^1}_{4\phi} = -\sinh(\psi)\cos(\chi)\sin(\theta), \tag{6} $$

$$ {\omega^1}_{3\phi} = -\sinh(\psi)\sin(\chi)\sin(\theta), \qquad {\omega^1}_{4\theta} = \sinh(\psi)\sin(\chi) \tag{7} $$

$$ {\omega^3}_{4\phi} = -\cos(\theta). \tag{8} $$

where $\chi = \chi(t,r)$ and $\psi = \psi(t,r)$ are arbitrary functions. The coordinates are $x^\mu = [t, r, \theta, \phi]$.

The spin connection ${\omega^a}_{b\mu}$ is defined in terms of an arbitrary Lorentz transformation ${\Lambda^a}_b$ through the equation

$$ {\omega^a}_{b\mu} = {(\Lambda^{-1})^a}_d \partial_\mu ({\Lambda^d}_b). \tag{9} $$

So we end up with a linear system of PDEs for each $\mu = [t, r, \theta, \phi]$. In matrix format, if we let W_μ be the matrix having entries ${\omega^a}_{b\mu}$, then the PDEs can be expressed as

$$ \partial_\mu \Lambda = \Lambda W_\mu \tag{10} $$

We want to know the value of the matrix Λ.

II. SPIN CONNECTION AND THE PROPER FRAME

Let the matrix Λ hold the components $\Lambda^a{}_b$=$\Lambda^a{}_b(t,r,\phi,\theta)$. We construct (10) for $\mu = [t,r,\theta,\phi]$ using W_μ and Λ. The null curvature condition must hold.

$$\partial_\mu W_\nu - \partial_\nu W_\mu + W_\mu W_\nu - W_\nu W_\mu = 0 \tag{11}$$

The Λ that generates the most general spherically symmetric spin connection includes a constant, non-singular matrix factor, C which is multiplied by three rotations and one boost in the following construction.

$$\Lambda = C \cdot R_1(\phi) \cdot R_3(\theta) \cdot R_1(\chi) \cdot B_1(\psi) \tag{12}$$

Where

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \qquad R_3(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$R_1(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\chi) & -\sin(\chi) \\ 0 & 0 & \sin(\chi) & \cos(\chi) \end{bmatrix} \qquad B_1(\psi) = \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ \sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

As of now, the functions defining the spin connection, $\psi(t,r)$ and $\chi(t,r)$ are arbitrary but may be chosen to generate a Λ with particular properties, certain values for $\psi(t,r)$ and $\chi(t,r)$ alter or nullify the other rotations and boosts. For example we can choose Λ so that the anti-symmetric part of the covariant $f(T)$ field equations is satisfied given the proper frame, $\mathbf{h}^a$. The general solution will be shown where $C = \mathbb{1}$.

$$\Lambda = \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ \cos(\theta)\sinh(\psi) & \cosh(\psi)\cos(\theta) & \cos(\chi)\sin(\theta) & -\sin(\theta)\sin(\chi) \\ -\sin(\theta)\cos(\phi)\sinh(\psi) & -\cos(\phi)\sin(\theta)\cosh(\psi) & \cos(\theta)\cos(\phi)\cos(\chi)-\sin(\phi)\sin(\chi) & -\cos(\phi)\cos(\theta)\sin(\chi)-\sin(\phi)\cos(\chi) \\ -\sin(\phi)\sin(\theta)\sinh(\psi) & -\sin(\phi)\sin(\theta)\cosh(\psi) & \cos(\theta)\sin(\phi)\cos(\chi)+\cos(\phi)\sin(\chi) & -\cos(\theta)\sin(\phi)\sin(\chi)+\cos(\phi)\cos(\chi) \end{bmatrix}$$

If we wish to move from the diagonal frame

$$\mathbf{h}^a = \begin{bmatrix} A_1\left(t,r\right)\mathbf{d}t \\ A_2\left(t,r\right)\mathbf{d}r \\ A_3\left(t,r\right)\mathbf{d}\theta \\ A_3\left(t,r\right)\sin\left(\theta\right)\mathbf{d}\phi \end{bmatrix} \tag{15}$$

to the proper spherically symmetric co-frame for a general teleparallel geometry, we can compute

$$\mathbf{h}^a = \Lambda^a{}_b\mathbf{h}^b \tag{16}$$

which will take the following form, up to a constant, non-singular matrix factor

$$\mathbf{h}^a = \begin{bmatrix} A_1(t,r)\cosh(\psi)\mathbf{d}t + A_2(t,r)\sinh(\psi)\mathbf{d}r \\ A_1(t,r)\sinh(\psi)\mathbf{d}t\cos(\theta) + A_2(t,r)\cosh(\psi)\mathbf{d}r\cos(\theta) + A_3(t,r)\cos(\chi)\mathbf{d}\theta\sin(\theta) - A_3(t,r)\sin(\chi)\mathbf{d}\phi\sin^2(\theta) \\ -A_1(t,r)\sinh(\psi)\cos(\phi)\mathbf{d}t\sin(\theta) - A_2(t,r)\cosh(\psi)\mathbf{d}r\cos(\phi)\sin(\theta) + A_3(t,r)\mathbf{d}\theta(cos(\theta)\cos(\phi)\cos(\chi) - \sin(\phi)\sin(\chi)) - A_3(t,r)\sin(\theta)\mathbf{d}\phi(cos(\phi)\cos(\theta)\sin(\chi) + sin(\phi)\cos(\chi)) \\ -A_1(t,r)\sinh(\psi)\mathbf{d}t\sin(\phi)\sin(\theta) - A_2(t,r)\cosh(\psi)\mathbf{d}r\sin(\phi)\sin(\theta) + A_3(t,r)\mathbf{d}\theta(cos(\theta)\sin(\phi)\cos(\chi) + \cos(\phi)\sin(\chi)) + A_3(t,r)\sin(\theta)\mathbf{d}\phi(-\cos(\theta)\sin(\phi)\sin(\chi) + cos(\phi)\cos(\chi)) \end{bmatrix}$$

where the co-frame components are

$$h^a{}_\mu = \begin{bmatrix} A_1(t,r)\cosh(\psi) & A_2(t,r)\sinh(\psi) & 0 & 0 \\ A_1(t,r)\cos(\theta)\sinh(\psi) & A_2(t,r)\cosh(\psi)\cos(\theta) & A_3(t,r)\cos(\chi)\sin(\theta) & -A_3(t,r)\sin^2(\theta)\sin(\chi) \\ -A_1(t,r)\sin(\theta)\cos(\phi)\sinh(\psi) & -A_2(t,r)\cos(\phi)\sin(\theta)\cosh(\psi) & A_3(t,r)(\cos(\theta)\cos(\phi)\cos(\chi) - \sin(\phi)\sin(\chi)) & -A_3(t,r)\sin(\theta)(\cos(\phi)\cos(\theta)\sin(\chi) + \sin(\phi)\cos(\chi)) \\ -A_1(t,r)\sin(\phi)\sin(\theta)\sinh(\psi) & -A_2(t,r)\sin(\phi)\sin(\theta)\cosh(\psi) & A_3(t,r)(\cos(\theta)\sin(\phi)\cos(\chi) + \cos(\phi)\sin(\chi)) & A_3(t,r)\sin(\theta)(-\cos(\theta)\sin(\phi)\sin(\chi) + \cos(\phi)\cos(\chi)) \end{bmatrix}$$

We describe the proper co-frame with 5 arbitrary functions of t and r. Another approach to the same problem has been taken by Hohmann et al. [2] .

$$\tilde{h}^a{}_\mu = \begin{bmatrix} C_1 & C_2 & 0 & 0 \\ C_3\sin(\theta)\cos(\phi) & C_4\cos(\phi)\sin(\theta) & C_5\cos(\theta)\cos(\phi) - C_6\sin(\phi) & -\sin(\theta)(C_5\sin(\phi) + C_6\cos(\theta)\cos(\phi)) \\ C_3\sin(\phi)\sin(\theta) & C_4\sin(\phi)\sin(\theta) & C_5\cos(\theta)\sin(\phi) + C_6\cos(\phi) & \sin(\theta)(C_5\cos(\phi) - C_6\cos(\theta)\sin(\phi)) \\ C_3\cos(\theta) & C_4\cos(\theta) & -C_5\sin(\theta) & C_6\sin^2(\theta) \end{bmatrix}$$

We will show that their solution is the same as our own, up to a constant matrix factor.

The six functions $C_i = C_i(t, r)$ $(i = 1, \ldots, 6)$ can depend on time and the radial coordinate. The line element in [2] is

$$\mathbf{d}s^2 = -(C_1^2 - C_3^2)\mathbf{d}t^2 + 2(C_3C_4 - C_1C_2)\mathbf{d}t\mathbf{d}r + (C_4^2 - C_2^2)\mathbf{d}r^2 + (C_5^2 + C_6^2)\mathbf{d}\Omega^2$$

where $\mathbf{d}\Omega^2 = \left(\mathbf{d}\theta^2 + \sin^2(\theta)\mathbf{d}\phi^2\right)$.

New coordinates can be chosen so that $C_1C_2 - C_3C_4 = 0$, which reduces Hohmann's number of arbitrary functions to 5. We start by comparing the line elements to solve for $C_i(t, r)$ in terms of our five arbitrary functions $A_1(t, r),\ A_2(t, r),\ A_3(t, r),\ \psi(t, r),\ \chi(t, r)$ so that Hohmann's metric will match ours, we will then find a constant matrix factor, λ^a_b which relates the two solutions.

$$\mathbf{d}s^2 = -A_1^2(t, r)\mathbf{d}t^2 + A_2^2(t, r)\mathbf{d}r^2 + A_3^2(t, r)\mathbf{d}\theta^2 + A_3^2(t, r)\sin^2(\theta)\mathbf{d}\phi^2$$

we find the following constraints,

$$C_1^2 - C_3^2 = A_1^2(t, r), \quad C_3C_4 - C_1C_2 = 0, \quad C_4^2 - C_2^2 = A_2^2(t, r), \quad C_5^2 + C_6^2 = A_3^2(t, r)$$

It is now rather straight forward to solve for $C_i(t, r)$,

$$C_1 = A_1(t, r)\cosh(\psi) \quad C_2 = A_2(t, r)\sinh(\psi) \quad C_3 = A_1(t, r)\sinh(\psi) \quad C_4 = A_2(t, r)\cosh(\psi)$$
$$C_5 = -A_3(t, r)\cos(\chi) \quad C_6 = -A_3(t, r)\sin(\chi)$$

substituting $C_i(t, r)$ into $\tilde{h}^a{}_\mu$ we find,

$$\tilde{h}^a{}_\mu = \begin{bmatrix} A_1(t,r)\cosh(\psi) & A_2(t,r)\sinh(\psi) & 0 & 0 \\ A_1(t,r)\sin(\theta)\cos(\phi)\sinh(\psi) & A_2(t,r)\cos(\phi)\sin(\theta)\cosh(\psi) & -A_3(t,r)(\cos(\chi)\cos(\theta)\cos(\phi) - \sin(\chi)\sin(\phi)) & A_3(t,r)\sin(\theta)(\cos(\chi)\sin(\phi) + \sin(\chi)\cos(\theta)\cos(\phi)) \\ A_1(t,r)\sin(\phi)\sin(\theta)\sinh(\psi) & A_2(t,r)\sin(\phi)\sin(\theta)\cosh(\psi) & -A_3(t,r)(\cos(\chi)\cos(\theta)\sin(\phi) + \sin(\chi)\cos(\phi)) & -A_3(t,r)\sin(\theta)(\cos(\chi)\cos(\phi) - \sin(\chi)\cos(\theta)\sin(\phi)) \\ A_1(t,r)\cos(\theta)\sinh(\psi) & A_2(t,r)\cosh(\psi)\cos(\theta) & A_3(t,r)\cos(\chi)\sin(\theta) & -A_3(t,r)\sin^2(\theta)\sin(\chi) \end{bmatrix}$$

The metrics are now the same however the co-frame is slightly different. Recall we can multiply Λ by any constant, non-singular matrix without any loss of generality. So, multiplying by the matrix $\lambda^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ we arrive at the expression

$$h^a{}_\mu = \lambda^a{}_b \tilde{h}^b{}_\mu \tag{17}$$

And so the same co-frame determined by Hohmann et al.[2] is equivalent to the proper frame which has been computed from the general spherically symmetric geometry in [1].

III. NATURAL SUB-CASES

From a general spherically symmetric, teleparallel geometry there are two natural sub-cases which are of some practical value. The isometry group of both sub-geometries is $SO(3) \times \mathbb{R}$.

- **STATIC SUB-CASE**

In the static sub-case each arbitrary function loses its dependence on t and we are left with a geometry which is invariant under time translation, adding another dimension to the symmetry group. In this construction it will be assumed that $\psi = \psi(r)$ and $\chi = \chi(r)$. We can choose coordinates so that $A_3(r) = r$ without any loss of generality in the static sub-case.

$$\mathbf{d}s^2 = -A_1^2(r)\mathbf{d}t^2 + A_2^2(r)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2(\theta)\mathbf{d}\phi^2$$

This is the general line element for a static spherically symmetric geometry, the co-frame becomes,

$$h^a{}_\mu = \begin{bmatrix} A_1(r)\cosh(\psi) & A_2(r)\sinh(\psi) & 0 & 0 \\ A_1(r)\cos(\theta)\sinh(\psi) & A_2(r)\cosh(\psi)\cos(\theta) & r\cos(\chi)\sin(\theta) & -r\sin^2(\theta)\sin(\chi) \\ -A_1(r)\sin(\theta)\cos(\phi)\sinh(\psi) & -A_2(r)\cos(\phi)\sin(\theta)\cosh(\psi) & r(\cos(\theta)\cos(\phi)\cos(\chi)-\sin(\phi)\sin(\chi)) & -r\sin(\theta)(\cos(\phi)\cos(\theta)\sin(\chi)+\sin(\phi)\cos(\chi)) \\ -A_1(r)\sin(\phi)\sin(\theta)\sinh(\psi) & -A_2(r)\sin(\phi)\sin(\theta)\cosh(\psi) & r(\cos(\theta)\sin(\phi)\cos(\chi)+\cos(\phi)\sin(\chi)) & r\sin(\theta)(-\cos(\theta)\sin(\phi)\sin(\chi)+\cos(\phi)\cos(\chi)) \end{bmatrix}$$

- **KANTOWSKI-SACHS SUB-CASE**

In the Kantowski-Sachs sub-case we use only arbitrary functions of the coordinate t, leading to a geometry invariant under radial translations. It will be assumed in this scenario that $\psi = \psi(t)$ and $\chi = \chi(t)$. We can choose coordinates such that $A_1(t) = 1$ without any loss of generality here. With the substitution we reduce to the standard line element for a general Kantowski-Sachs geometry.

$$\mathbf{d}s^2 = -\mathbf{d}t^2 + A_2^2(t)\mathbf{d}r^2 + A_3^2(t)\mathbf{d}\theta^2 + A_3^2(t)\sin^2(\theta)\mathbf{d}\phi^2$$

And the co-frame becomes

$$h^a{}_\mu = \begin{bmatrix} \cosh(\psi) & A_2(t)\sinh(\psi) & 0 & 0 \\ \cos(\theta)\sinh(\psi) & A_2(t)\cosh(\psi)\cos(\theta) & A_3(t)\cos(\chi)\sin(\theta) & -A_3(t)\sin^2(\theta)\sin(\chi) \\ -\sin(\theta)\cos(\phi)\sinh(\psi) & -A_2(t)\cos(\phi)\sin(\theta)\cosh(\psi) & A_3(t)(\cos(\theta)\cos(\phi)\cos(\chi)-\sin(\phi)\sin(\chi)) & -A_3(t)\sin(\theta)(\cos(\phi)\cos(\theta)\sin(\chi)+\sin(\phi)\cos(\chi)) \\ -\sin(\phi)\sin(\theta)\sinh(\psi) & -A_2(t)\sin(\phi)\sin(\theta)\cosh(\psi) & A_3(t)(\cos(\theta)\sin(\phi)\cos(\chi)+\cos(\phi)\sin(\chi)) & A_3(t)\sin(\theta)(-\cos(\theta)\sin(\phi)\sin(\chi)+\cos(\phi)\cos(\chi)) \end{bmatrix}$$

- **EINSTEIN TELEPARALLEL SUB-CASE**

It will be assumed that $z = z(t, r)$ $\psi = \psi(z(t, r))$ and $\chi = \chi(z(t, r))$. We will choose coordinates so that $\frac{A_3(z(t,r))}{r} = A_2(z(t, r))$ without any loss of generality in the einstein teleparallel sub-case.[2]

$$\mathbf{d}s^2 = -A_1^2(z(t,r))\mathbf{d}t^2 + A_2^2(z(t,r))(\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2(\theta)\mathbf{d}\phi^2)$$

the co-frame components become

$$h^a{}_\mu = \begin{bmatrix} A_1(z)\cosh(\psi) & A_2(z)\sinh(\psi) & 0 & 0 \\ A_1(z)\cos(\theta)\sinh(\psi) & A_2(z)\cosh(\psi)\cos(\theta) & A_2(z)r\cos(\chi)\sin(\theta) & -A_2(z)r\sin^2(\theta)\sin(\chi) \\ -A_1(z)\sin(\theta)\cos(\phi)\sinh(\psi) & -A_2(z)\cos(\phi)\sin(\theta)\cosh(\psi) & A_2(z)r(\cos(\theta)\cos(\phi)\cos(\chi) - \sin(\phi)\sin(\chi)) & -A_2(z)r\sin(\theta)(\cos(\phi)\cos(\theta)\sin(\chi) + \sin(\phi)\cos(\chi)) \\ -A_1(z)\sin(\phi)\sin(\theta)\sinh(\psi) & -A_2(z)\sin(\phi)\sin(\theta)\cosh(\psi) & A_2(z)r(\cos(\theta)\sin(\phi)\cos(\chi) + \cos(\phi)\sin(\chi)) & A_2(z)r\sin(\theta)(-\cos(\theta)\sin(\phi)\sin(\chi) + \cos(\phi)\cos(\chi)) \end{bmatrix}$$

IV. Notes:

It is not yet clear whether we can make the substitution $A_3(t, r) = r$ in general, in theory a spherically symmetric metric need only two arbitrary functions to be fully defined, however the symmetries present in the co-frame components don't necessarily carry over to the metric.

REFERENCES

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