Inverse Functions

To demonstrate the idea of inverse functions, let's start with a simple example.

Example 1: Suppose that $y(x) = 3x + 5$ gives the distance (in m) that you have travelled by time x (in sec). There are two simple questions that can be asked:

1. How far have you gone after 7 sec?
2. How long did it take you to travel 26 m?

Answer to 1: This is the common one, we substitute __X__ = __7__ and get $y = 3(7)+5 = 26$ m

So, in this problem, we knew the value of __X__ and had to find the value of __Y__.

Answer to 2: This is the inverse problem because we now know __Y__ and we must find __X__. If you look at part 1, we know the answer is __X__ = __7__.

That was lucky since, in part 1, we did the opposite and we could basically just copy the answer. But there are (in this case) **two other ways (both involve simple algebra)**.

(a) We could substitute __Y__ = __26__ in the original formula and solve for __X__:

$26 = 3x+5 \Rightarrow 21 = 3x \Rightarrow \boxed{x = 7}$

(b) We could find a formula that undoes $y(x)$ and gives x as a function of y (that is, solve for x):

$Y = 3x+5 \quad 3x = y-5 \Rightarrow \boxed{x = \frac{y-5}{3}}$

Now, you can use a similar method as in question 1 and just substitute __Y__ = __26__ into this formula:

$x = \frac{26-5}{3} = 7.$

Question: How much time will it take to go 100 m?

$x = \frac{100-5}{3} = \frac{95}{3}$

Theory of Inverses

Question # 2 above asked the inverse of question # 1, particularly:

- In question 1, input = __time, x__ and output = __dist Y__
- In question 2, input = __dist Y__ and output = __time x__

For obvious reasons, these processes are called **inverses**, and in question 2 we found the inverse function of $y(x)$ (that is, the function that undoes $y(x)$). This gives the following obvious but important observation:

$\boxed{f^{-1}(x) \neq \frac{1}{f(x)}}$

The inverse function of $f(x)$, usually denoted as $f^{-1}(x)$, reverses the roles of input and output. Therefore, in the inverse function, x becomes y and y becomes x.

This seems rather simple, and it is – use simple algebra to solve for x in terms of y. The only thing is that we usually use x as the input and y for the output, and we graph x on the horizontal axis and y on the vertical. For this reason, after having found the inverse formula, we interchange the names of x and y.

Procedure for Finding an Inverse Function (if possible):

1. If the function is named $f(x)$, re-label it as y.
2. Solve for x in terms of y.
3. Interchange the names of x and y. The result is the inverse function $f^{-1}(x)$.

Example 2: (a) Find the inverse of $f(x)=8x-4$. (b) Find $f(2)$ and $f^{-1}(12)$. (c) Find $f^{-1}(f(x))$.

a) $y=8x-4$

$8x=y+4$

$x=\frac{y+4}{8}$

$\Rightarrow y=\frac{x+4}{8}$

$\boxed{f^{-1}(x)=\frac{x+4}{8}}$

b) $f(2)=8(2)-4=12$

$f^{-1}(12)=\frac{12+4}{8}=2$

c) $f^{-1}(8x-4)$

$=\frac{(8x-4)+4}{8}=\frac{8x}{8}=x$

This example actually shows two logical rules that apply to all inverses. Note the following:

1. $f(\overset{x}{2})=\overset{y}{12}$ and $f^{-1}(\overset{y}{12})=\overset{x}{2}$.
   This gives rule # 1 of inverses: If $f(x)=y$ then $x=f^{-1}(y)$.

$f(x)=y$

$f^{-1}(f(x))=f^{-1}(y)$

$x=f^{-1}(y)$

2. In part (c) we found that $f^{-1}(f(x))=x$ (as expected, since $f^{-1}(x)$ is supposed to "undo" what $f(x)$ has done to x). The reverse composition is also true.
   This gives rule # 2 of inverses: If $f^{-1}(x)$ is the inverse of $f(x)$, then $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$.

Domain and Range of Inverses

This is another very logical result coming from the fact that the roles of x and y (input and output) are reversed.

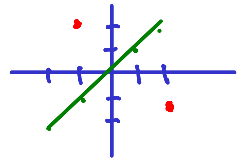
- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.

Graphing Inverses

Again, this is simple if you use the central idea that the roles of x and y are switched. If on the graph of $f(x)$ the point (a, b) is plotted, then the graph of $f^{-1}(x)$ contains the point (b,a). In other words:

- If you have the graph of $f(x)$, you can obtain the graph of $f^{-1}(x)$ by reflecting $f(x)$ about the line $y=x$.

Example 3: To convince yourself of this fact, suppose the point (-1, 2) is on the graph of $f(x)$. Then the point $(2,-1)$ will be on the graph of $f^{-1}(x)$. Plot these points:



Now, draw the line $y = x$ on that graph. You can see that the second point is just the reflection of the first through this line. Since this works for all points on a graph, then the entire graph of $f^{-1}(x)$ can be generated by reflecting $f(x)$ through this line.

Example 4: Suppose $f(x) = \sqrt{x+3}$.

$x+3 \geq 0$

$x \geq -3$

(a) Find the domain and range of $f(x)$ and $f^{-1}(x)$.

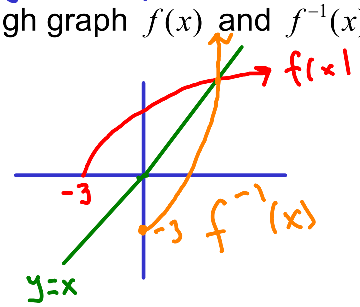
$f(x)$ $D = \{x \geq -3\}$

$R = \{y \geq 0\}$

$f^{-1}(x)$ $D = \{x \geq 0\}$ (Range of $f(x)$)

$R = \{y \geq -3\}$ (Domain of $f(x)$)

(b) Sketch a rough graph $f(x)$ and $f^{-1}(x)$.



(c) Find a formula for $f^{-1}(x)$.

$y = \sqrt{x+3}$

$y^2 = x+3$

$x = y^2 - 3$

$y = x^2 - 3$

$\Rightarrow \boxed{f^{-1}(x) = x^2 - 3}$

More Theory of Inverses

Example 5: Let $f(x) = x^2$. Can you find a single function $f^{-1}(x)$?

$y = x^2$ $x = \pm\sqrt{y}$ $f^{-1}(x) = \pm\sqrt{x}$ $\rightarrow$ Two Functions: $\sqrt{x}$, $-\sqrt{x}$

This is actually not possible for inverses. **A function MUST have ONLY ONE inverse!** The problem with $f(x) = x^2$ is that two values of x (i.e., more than one) produce each value of y. For example,

- If y = 9, then x = 3 and x = −3 produce it.
- In other words, $f(3) = 9$ and $f(-3) = 9$.

So that if we now ask for the value of $f^{-1}(9)$, we can't give a definite answer since there is more than one possibility. This gives the following very important property of inverse functions:

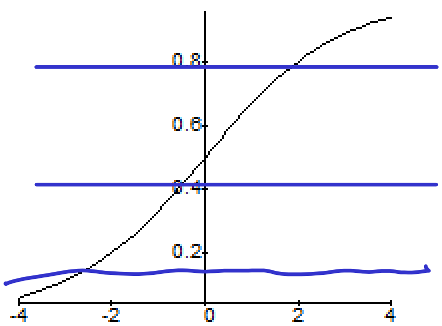
For a function to have an inverse, two different inputs x are NOT permitted to produce the same value of y. This property is called One to one (often abbreviated as 1:1).

1. Algebraically: A function $f(x)$ is one – to – one if, for different values of the input x (say, x_1 and x_2), we have $f(x_1) \neq f(x_2)$. For any pair of values x_1, x_2

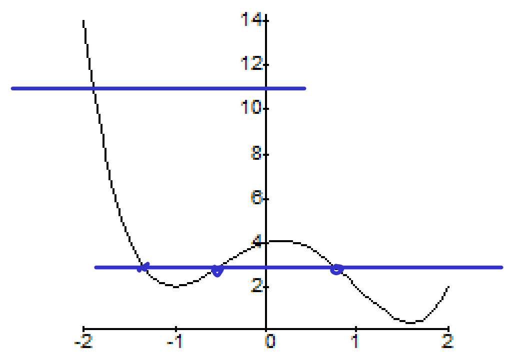
2. Graphically: No two y values can be the same, so if the graph hits a particular value of y, it cannot turn around and come back to that value. That is, it can never achieve the same height above the x – axis twice. That means the graph is either always increasing or always decreasing. There are NO turning points! Note that this can also be thought of as a horizontal line test: If the graph of $f(x)$ never intersects any horizontal line more than once, then $f(x)$ is one – to – one.

Example 6: In each part below, a function is graphed. Determine whether the function has an inverse.

a. Has an inverse



b. No inverse



Putting all the observations and theories together gives the precise definition of an inverse.

Precise Definition of an Inverse: Let $f(x)$ be a 1:1 function with domain A and range B. Then its inverse function, $f^{-1}(x)$, has domain B and range A, and is defined by

$f(x) = y$ if and only if $f^{-1}(y) =$ x for every value of y in B.