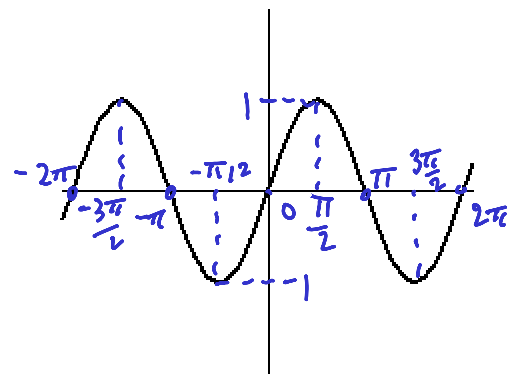
Inverse Trigonometric Functions

The function $f(x) = \sin(x)$ is graphed below (fill in some important points):



Does this function have an inverse? No Why? Fails HLT

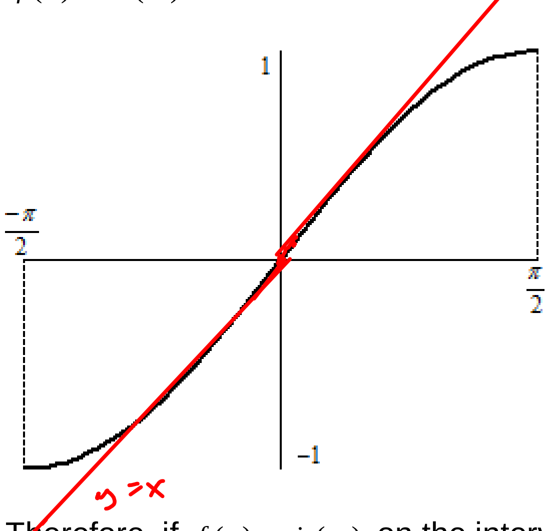
It is very often useful to have inverses for commonly used functions (such as $f(x) = \sin(x)$). The inverse of this function may not exist, but we may achieve an inverse by <u>restricting the domain</u> of $f(x) = \sin(x)$ so that the following are accomplished:

a. We hit every value in its range, which is $[-1, 1]$.
b. We do not repeat any of these values (so that we get a 1:1 function).

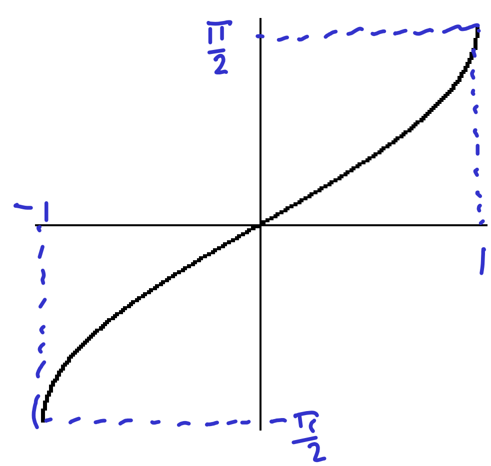
Give some domain restrictions that accomplishes these tasks: $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, ...
Which one is correct? They are actually all correct, but by convention, we'll always use $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now we can graph $f(x) = \sin(x)$ on that restricted domain. We can now also graph its inverse by reflecting about the line $y = x$.

$f(x) = \sin(x)$ on restricted domain:



Inverse of $f(x) = \sin(x)$, called $\sin^{-1}(x)$ or $\arcsin(x)$



Therefore, if $f(x) = \sin(x)$ on the interval $[-\pi/2, \pi/2]$, then $f^{-1}(x) = \arcsin(x)$.

$D = [-1, 1]$, $R = [-\pi/2, \pi/2]$

Example 1: Find $\arcsin\left(\frac{1}{\sqrt{2}}\right)$. (Thought of differently, we want the angle x such that $\sin(x) = \frac{1}{\sqrt{2}}$).

$$\sin(\pi/4) = \frac{1}{\sqrt{2}} \Rightarrow \arcsin\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\pi}{4}}$$

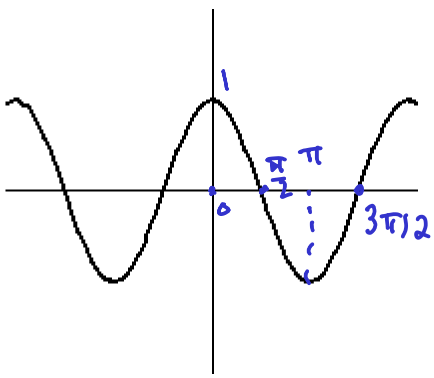
Note: There are actually a lot of angles x such that $\sin(x) = \frac{1}{\sqrt{2}}$: $3\pi/4,\ 9\pi/4,\ 11\pi/4, \dots$.

Are all these correct as well? NO. Why?

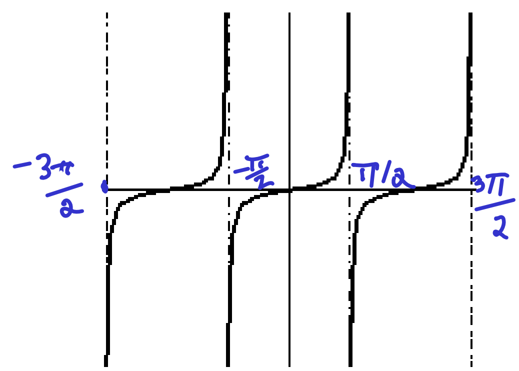
Range of $\arcsin(x)$ is $[-\pi/2, \pi/2]$

A similar thing can be done to obtain inverses for the other trigonometric functions. We will only study $\cos(x)$, $\tan(x)$, and $\sec(x)$. Here are their graphs, respectively (again, fill in important points):

Graph of $f(x) = \cos(x)$

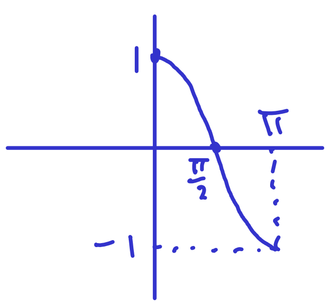


Graph of $f(x) = \tan(x)$



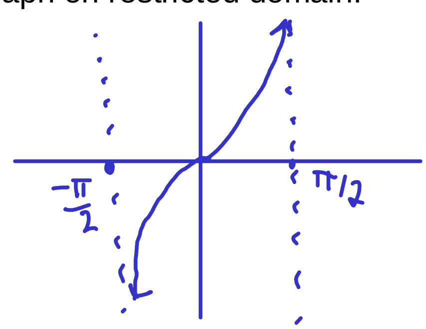
Domain restriction for $\cos(x)$ $[0, \pi]$

Graph on restricted domain:

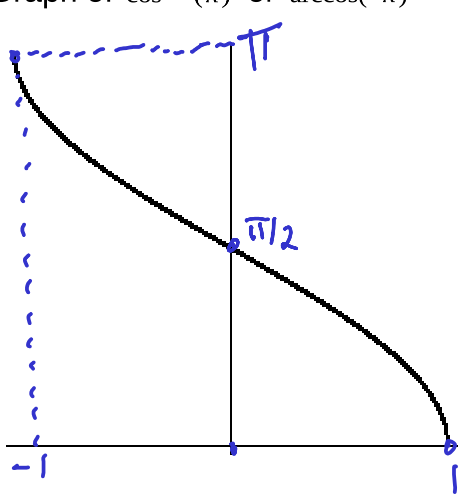


Domain restriction for $\tan(x)$ $(-\pi/2, \pi/2)$

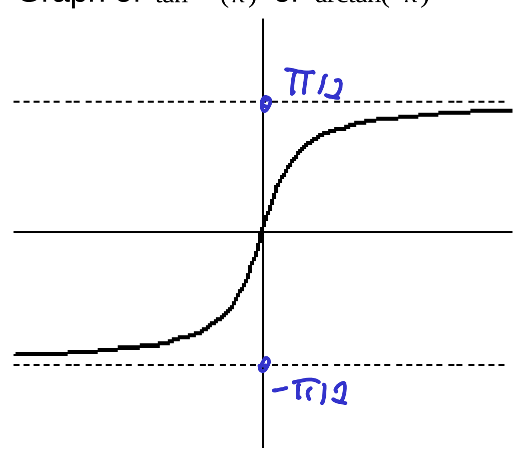
Graph on restricted domain:



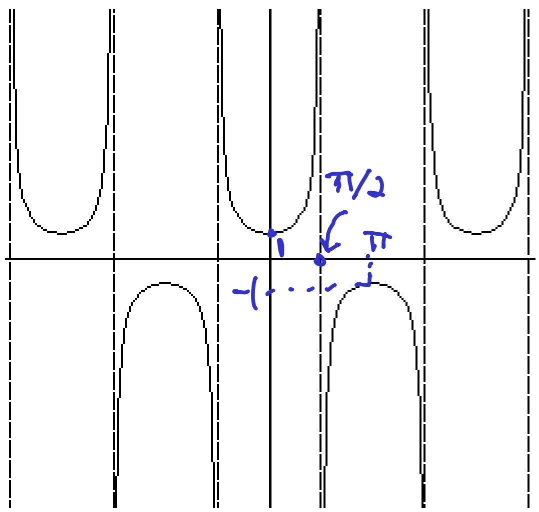
Graph of $\cos^{-1}(x)$ or $\arccos(x)$



Graph of $\tan^{-1}(x)$ or $\arctan(x)$

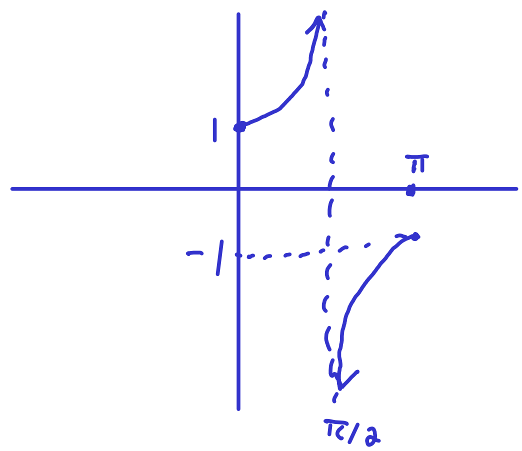


Graph of $f(x) = \sec(x)$

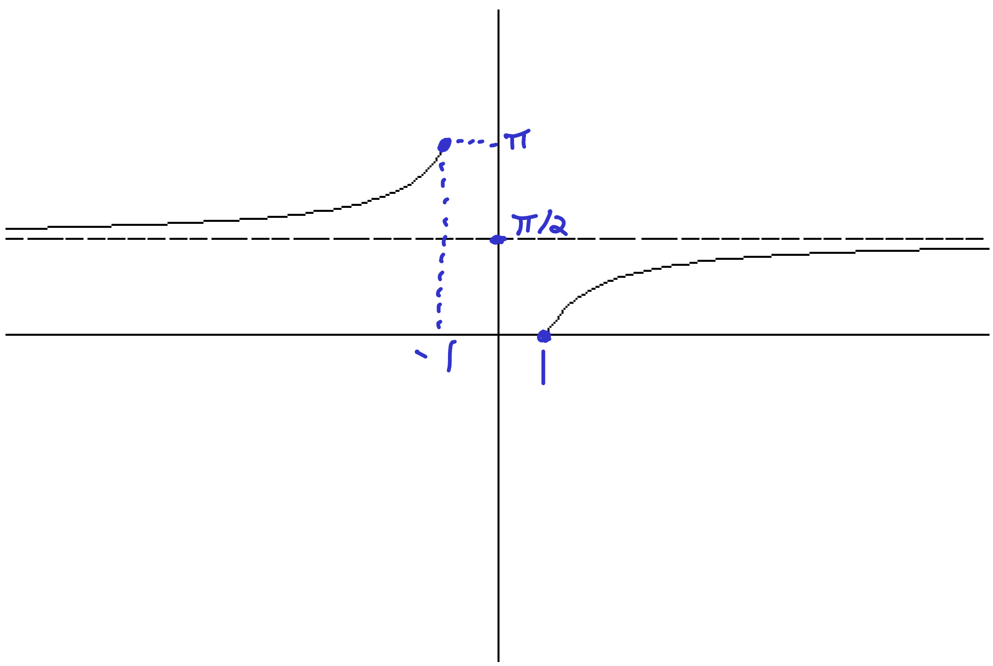


Domain restriction for sec(x) $[0, \pi/2) \cup (\pi/2, \pi]$

Graph on restricted domain:



Graph of $\sec^{-1}(x)$ or $\operatorname{arc\,sec}(x)$



$D = (-\infty, -1] \cup [1, \infty)$

$R = [0, \pi/2) \cup (\pi/2, \pi]$

Summary of Inverse Trigonometric Function Properties

$f(f^{-1}(x)) \,,\, f^{-1}(f(x)) = x$

	$\sin^{-1}(x)$ or $\arcsin(x)$	$\cos^{-1}(x)$ or $\arccos(x)$	$\tan^{-1}(x)$ or $\arctan(x)$	$\sec^{-1}(x)$ or $\text{arc}\sec(x)$
Domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Range	$[-\pi/2, \pi/2]$	$[0, \pi]$	$(-\pi/2, \pi/2)$	$[0, \pi/2) \cup (\pi/2, \pi]$
Cancellation	$\sin(\sin^{-1}(x)) = x$ on $[-1,1]$. $\sin^{-1}(\sin(x)) = x$ on $[-\pi/2, \pi/2]$	$\cos(\cos^{-1}(x)) = x$ on $[-1,1]$. $\cos^{-1}(\cos(x)) = x$ on $[0,\pi]$.	$\tan(\tan^{-1}(x)) = x$ on ______. $\tan^{-1}(\tan(x)) = x$ on ______.	$\sec(\sec^{-1}(x)) = x$ on ______. $\sec^{-1}(\sec(x)) = x$ on ______.

Example 2: Find the following:

(a) $\arccos(\sqrt{3}/2)$ $\cos(\pi/6) = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\boxed{\arccos\left(\frac{\sqrt{3}}{2}\right) = \pi/6.}$

(b) $\sin^{-1}(2)$ Does not exist. 2 not in domain of arcsin.

(c) $\arcsin(\sin(0.31))$ $= 0.31$

(d) $\arcsin(\sin(2.38))$ $\neq 2.38$. $\rightarrow$ Because 2.38 is not in Range of arcsin.

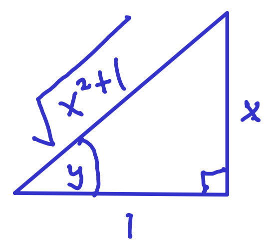
(e) $\sin(\arcsin(2.38))$ $\rightarrow$ Does not exist. 2.38 not in domain of arcsin.

Example 3: Simplify the function $\sin(\arctan(x))$ into algebraic form.

Let $y = \arctan(x)$ Want: $\sin(y)$

$\Rightarrow \tan(y) = \cancel{\tan}(\arctan(x))$

$\boxed{\tan(y) = x}$. $\tan(y) = \frac{opp}{adj} \rightarrow \frac{x}{1}$



$\sin(\arctan(x)) = \sin(y) = \frac{opp}{hyp} = \boxed{\frac{x}{\sqrt{x^2+1}}}$